Supersymmetric Quantum Mechanics and Morse theory

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Lectures 16-18: Quantum Mechanics v0

Contents

1 Quantum Mechanics

1.1 From topology to quantum mechanics

In this section I want to briefly outline the structure of the second half of the course. Let us recall some results about the topology:

- Topology is a shape of a smooth manifold, which we can describe using topological invariants.
- De Rham cohomology is one of the most popular and useful topological invariants.
- Hodge theorem relates de Rham cohomology to the harmonic forms.
- $H^0(\mathbb{T}^3)$ is in one to one corresponence with solutions

$$\Delta f = \sum_{i=1}^{3} \partial_i^2 f = 0 \tag{1.1}$$

Let us recall some basis results from quantum mechanics

- The wave-function$\Psi(x, y, z, t)$ of point particle moving in $\mathbb{R}^3$ obeys the Schrodinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + V(x, y, z)\right]\Psi \tag{1.2}$$

- We can consider stationary solutions

$$\Psi(x, y, z, t) = \Psi_E(x, y, z)e^{-\frac{i}{\hbar}Et} \tag{1.3}$$

with stationary wave-function $\Psi_E(x, y, z)$ obeying the stationary Schrodinger equation

$$\left[-\sum_{i=1}^{3}\frac{\hbar^2}{2m}\partial_i^2 + V(x)\right]\Psi_E = E\Psi_E \tag{1.4}$$

- We assume zero potential

$$-\sum_{i=1}^{3}\frac{\hbar^2}{2m}\partial_i^2\Psi_E = E\Psi_E \tag{1.5}$$

- The $E = 0$ stationary wavefunctions obey

$$-\frac{\hbar^2}{2m}\sum_{i=1}^{3}\partial_i^2\Psi_0 = 0 \tag{1.6}$$

Observation: The $H^0(\mathbb{T}^3)$ is in one to one correspondence with the $E = 0$ wavefunctions for the free particle in $\mathbb{T}^3$.

Outline of the second part of the course

1. Review the derivation of the Schrodinger equation for point particle in $\mathbb{R}^3$ and $\mathbb{T}^3$.
2. Generalize to $H^0(\mathbb{T}^n)$.
3. Include Grassmann variables to generalize to $H^k(\mathbb{T}^n)$
4. Generalize to $H^k(M)$
5. Add potential and relate to Morse theory
6. Use SUSY localization to simplify $H^k(M)$

1.2 Stationary action principle

Classical mechanics is about describing the trajectory $x(t)$ of a particle as a function of time t. In most cases trajectory is a solution to the second order differential equation

$$ma = m\ddot{x} = \frac{d^2x}{dt^2} = F(x) = -\partial_x V = -V' \tag{1.7}$$

known as a *Newton's law*. We need to supplement the second order differential equation with initial data: initial position $x(0)$ and velocity $\dot{x}(0)$.

Theoretical physics describes the same system using *stationary action principle*: Let us consider a functional on the space of trajectories $\mathcal{T} = C^\infty Map(I, \mathbb{R})$

$$S : \mathcal{T} \to \mathbb{R}, \quad S[x(t)] = \int_I L(x, \dot{x}) dt = \int_I dt \, \left(\frac{1}{2} m\dot{x}^2 - V(x) \right), \tag{1.8}$$

with L known as a *Lagrangian*. Let us choose $I = [0, 1]$. The equations of motion describe trajectories $x(t)$, that extremize the action at fixed boundary values

$$x(0) = x_I, \quad x(1) = x_F, \quad \delta x(0) = \delta x(1) = 0. \tag{1.9}$$

The variation of the action

$$\begin{aligned}
\delta S = S[x(t) + \delta x(t)] - S[x(t)] &= \int_0^1 dt(m\dot{x}\delta\dot{x} - V'\delta x) \\
&= \int_0^1 dt(\partial_t(m\dot{x}\delta x) - m\partial_t(\dot{x})\delta x - V'\delta x) \\
&= m\dot{x}\delta x\Big|_0^1 - \int_0^1 dt(m\ddot{x} + V')\delta x \\
&= m\dot{x}(1)\delta x(1) - m\dot{x}(0)\delta x(0) - \int_0^1 dt(m\ddot{x} + V')\delta x \\
&= -\int_0^1 dt(m\ddot{x} + V')\delta x
\end{aligned} \tag{1.10}$$

vanishes for arbitrary trajectory variation $\delta x(t)$ if and only if

$$m\ddot{x}(t) + V'(x(t)) = 0 \tag{1.11}$$

holds for all $t \in [0, 1]$, what matches with the Newton's law.

In case of general Lagrangian $L(x, \dot{x})$ the extremum of the action is on trajectories which solve

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \tag{1.12}$$

The equation above is kown as the *Euler-Lagrange equation*

1.3 Phase space

We can turn the second order differential equation into pair of the first order ones:

$$p(t) = m\dot{x}(t), \quad \dot{p}(t) = -V'(x(t)) \tag{1.13}$$

We can derive these equations from *first order stationary action principle*

$$S[x,p] = \int_I dt \left(p\dot{x} - \frac{p^2}{2m} - V(x) \right), \tag{1.14}$$

The boundary conditions for variations are

$$x(0) = x_I, \;\; x(1) = x_F, \;\; \delta x(0) = \delta x(1) = 0. \tag{1.15}$$

while p is unconstrained.

In case of generic Euler-Lagrange equation the corresponding first order equations

$$\dot{p} = -\frac{\partial H}{\partial x}, \quad \dot{x} = \frac{\partial H}{\partial p} \tag{1.16}$$

are known as the *Hamilton's equations* with *Hamiltonian* $H(p,x)$ being Legendre transform of the Lagrangian $L(x,\dot{x})$

$$H(p,x) = p\dot{x} - L(x,\dot{x}), \;\; p = \frac{\partial L}{\partial \dot{x}}. \tag{1.17}$$

The p, x are coordinates on the *phase space of one-dimensional point particle*. The phase space is $\mathbb{R}^2$ endowed with a non-degenerate two form

$$\omega = dp \wedge dx \in \Omega^2(\mathbb{R}^2) \tag{1.18}$$

what makes a pair $(\mathbb{R}^2, \omega)$ into a *symplectic manifold*.

Definition: A pair (M, ω) defines a *symplectic* manifold iff M is an even dimensional and two-form $\omega \in \Omega^2(M)$ is such that

- ω is closed form i.e.

$$d\omega = 0, \tag{1.19}$$

- ω is non-degenerate form, i.e.

$$\forall p \in M \;\; \forall \xi \neq 0 \;\; \exists \eta : \;\; \omega(p, \xi, \eta) \neq 0, \;\; \xi, \eta \in T_pM. \tag{1.20}$$

Corollary: For closed compact M of dimension $2n = \dim(M)$, then we can define a nonvanishing $2n$ form $\omega^n \in \Omega^{2n}(M)$, which is the volume form i.e.

$$\int_M \omega^n = \mathrm{Vol}(M) \neq 0 \tag{1.21}$$

Being top form ω^n is a closed form, while it cannot be exact because integral above is nonzero. Thus we conclude that

$$\omega^n \in H^{2n}_{dR}(M) \neq 0 \tag{1.22}$$

Moreover, lower powers ω^k are also nontrivial in cohomology. By construction they are closed forms

$$d\omega^k = k d\omega \wedge \omega^{k-1} = 0. \tag{1.23}$$

Suppose ω^k is exact i.e.

$$\omega^k = d\alpha \tag{1.24}$$

then we can multiply it by ω^{n-k} and arrive into contradiction

$$0 \neq \mathrm{Vol}(M) = \int_M \omega^n = \int_M \omega^k \wedge \omega^{n-k} = \int_M d\alpha \wedge \omega^{n-k} = \int_M d(\alpha \wedge \omega^{n-k}) = 0. \tag{1.25}$$

Let us describe several useful examples of symplectic manifolds:

- The $(\mathbb{R}^2, \Omega_0)$ is a trivial example. The form is trivially closed $d\Omega_0 = d(dx \wedge dy) = 0$ and non-degenerate since $\omega_{xy} = 1$.
- Let X be a smooth manifold and T^*X being the cotangent bundle, then there is a canonical symplectic form $\omega_{can} = \sum dp_i \wedge dx^i$ that makes (T^*X, ω_{can}) into symplectic manifold.

- Let U be an open subset in a symplectic manifold (M, ω) then $(U, \omega|_U)$ is a symplectic manifold.
- The round unit sphere S^2 with coordinates (ϕ, θ) and the standard volume form

$$\omega_{S^2} = \sin\theta \; d\theta \wedge d\phi$$

  makes a pair (S^2, ω_{S^2}) into symplectic manifold.
- Let Σ be any oriented surface and ω any volume form on Σ then (Σ, ω) is a symplectic manifold. In particular ω is closed since $\dim(\Sigma) = 2$ and it is non-degenerate due to being a volume form.
- The S^4 is not a symplectic manifold. Given a properties of symplectic form we immediately conclude that the symplectic manifold should have nontrivial $H^{2k}_{dR}(M)$ In particular for the 4d manifold there should be $H^2_{dR}(M) \neq 0$ while in case of S^4 we have $H^2_{dR}(S^4) = 0$!

1.4 Canonical transformations

Definition: Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds. A smooth map $f : M_1 \to M_2$ is called a *symplectomorphism* (or a *canonical transformation*) if it is a diffeomorphism and

$$f^*\omega_2 = \omega_1. \tag{1.26}$$

We can use the symplectomorphism to locally represent any symplectic manifold as $(\mathbb{R}^{2n}, \Omega_0)$.

Theorem (Darboux): Let (M, ω) be a symplectic manifold of dimension $2n$. Then form any $p \in M$, there exists a neighborhood $p \in U \subset M$ and a neighborhood $p \in U_0 \subset \mathbb{R}^{2n}$ so that (U, ω) is symplectomorphic to (U_0, Ω_0), where Ω_0 is the standard symplectic form on $\mathbb{R}^{2n}$.

Corollary: A smooth map $f : M \to M$ is *symplectomorphism* (or a *canonical transformation*) if it is a diffeomorphism and it preserves the symplectic form ω on M i.e.

$$f^*\omega = \omega. \tag{1.27}$$

Definition: Let M be a smooth manifold and $\mathrm{Vect}(M)$ the set of all smooth vectors on M.

We can turn Vect(M) into Lie algebra if we define a Lie bracket

$$[\cdot,\cdot]:\text{Vect}(M)\times\text{Vect}(M)\to\text{Vect}(M),\quad (X,Y)\mapsto[X,Y]=XY-YX \tag{1.28}$$

that is bilinear, antisymmetric and obeys Jacobi identity

$$[[X,Y],Z]+[[Z,X],Y]+[[Y,Z],X]=0. \tag{1.29}$$

Thus a pair (Vect$(M),[\cdot,\cdot]$) is an (infinite dimensional) Lie algebra. The corresponding Lie group is the topologically trivial (contractible to the identity) diffeomorphisms Diff$^0(M)$ of M

$$\text{Diff}^0(M)=\{\phi:M\to M|\phi \text{ is a diffemorphism}\}. \tag{1.30}$$

We can define a group of symplectomorhisms Symp(M,ω) for a symplectic manifold (M,ω) as subgroup of diffeormorphisms

$$\text{Symp}(M,\omega)=\{\phi:M\to M|\phi^*\omega=\omega,\phi \text{ is a diffemorphism}\} \tag{1.31}$$

The last statement makes Symp(M,ω) into a proper subgroup of Diff$^0(M)$, while both are typically infinite dimensional.

1.5 Hamiltonian vector fields

The Lie algebra of Symp(M,ω) is a sub-algebra of (Vect$(M),[\cdot,\cdot]$) that includes vector fields that preserve ω

$$\text{symp}(M,\omega)=\{\mathcal{L}_\xi\omega=0,\quad \xi\in\text{Vect}(M)\} \tag{1.32}$$

what is straightforwardly follows from the Lie derivative property

$$\mathcal{L}_{[\xi,\eta]}\omega=\mathcal{L}_\xi\mathcal{L}_\eta\omega-\mathcal{L}_\eta\mathcal{L}_\xi\omega. \tag{1.33}$$

Given a symplectic vector field ξ we can use the Cartan formula for a Lie derivative and closeness of ω to simplify

$$0=\mathcal{L}_\xi\omega=(\iota_\xi\delta+\delta\iota_\xi)\omega=\delta(\iota_\xi\omega) \tag{1.34}$$

Locally we can write any closed form as an exact form i.e.

$$\iota_\xi \omega = \delta H_\xi \tag{1.35}$$

while globally the nontrivial $H^1(M)$ prevents from doing so. Thus we can define the *Hamiltonian vector field* as the vector field ξ such that there is a global function H_ξ. Straightforward check

$$\begin{aligned}\iota_{[\xi,\eta]}\omega &= (\mathcal{L}_\xi \iota_\eta - \iota_\eta \mathcal{L}_\xi)\omega = (\mathcal{L}_\xi \delta H_\eta - \iota_\eta \mathcal{L}_\xi \omega) = (\delta\iota_\xi + \iota_\xi \delta)\delta H_\eta \\ &= \delta\iota_\xi \delta H_\eta = \delta(\iota_\xi \delta + \delta\iota_\xi)H_\eta = \delta\mathcal{L}_\xi H_\eta = \delta(\mathcal{L}_\xi H_\eta)\end{aligned} \tag{1.36}$$

implies that the Hamiltonian vector fields form a Lie algebra $\mathrm{ham}(M,\omega)$.

In a language of exact sequences the relation between symplectic and hamiltonian vector fields.

$$0 \to \mathrm{ham}(M,\omega) \to \mathrm{symp}(M,\omega) \to H^1(M) \to 0 \tag{1.37}$$

Using symplectic form non-degeneracy of ω we can "invert" so for arbitrary function H on M there is a vector field X_H

$$H \mapsto X_H, \;\; dH = \iota_{X_H}\omega \tag{1.38}$$

1.6 Poisson structure

We can use ω to turn $C^\infty(M)$ into Lie algebra with *Poisson bracket*

$$\{G,H\} = \iota_{X_H}\iota_{X_G}\omega = \iota_{X_H} dG = \mathcal{L}_{X_H} G \tag{1.39}$$

Proposition: The Lie algebra $C^\infty(M)$ in general is a central extension of $\mathrm{ham}(M)$ i.e.

$$\{H_\eta, H_\xi\} = H_{[\eta,\xi]} + c(\eta,\xi) \tag{1.40}$$

Proof: We can verify the expression above explicitly

$$\begin{aligned} d\{H_\eta, H_\xi\} &= d\iota_\eta\iota_\xi\omega = \tfrac{1}{2}d\iota_\eta\iota_\xi\omega - \tfrac{1}{2}d\iota_\xi\iota_\eta\omega = \tfrac{1}{2}d\iota_\eta dH_\xi - \tfrac{1}{2}d\iota_\xi dH_\eta \\ &= \tfrac{1}{2}(d\iota_\eta + \iota_\eta d)dH_\xi - \tfrac{1}{2}(d\iota_\xi + \iota_\xi d)dH_\eta = \tfrac{1}{2}\mathcal{L}_\eta dH_\xi - \tfrac{1}{2}\mathcal{L}_\xi dH_\eta \\ &= \tfrac{1}{2}(\mathcal{L}_\eta\iota_\xi - \mathcal{L}_\xi\iota_\eta)\omega = \iota_{[\eta,\xi]}\omega = dH_{[\eta,\xi]}\end{aligned} \tag{1.41}$$

1.7 Symmetries

An infinitesimal action of the Lie algebra $\mathfrak{g}$ is a Lie-algebra morphism

$$\mathfrak{g} \to \mathrm{Vec}(M): \quad \epsilon \mapsto V_\epsilon, \quad V_{[\epsilon,\eta]} = [V_\epsilon, V_\eta] \tag{1.42}$$

Similarly we can define a Hamiltonian action as Lie algebra morphism

$$\mathfrak{g} \to \mathrm{ham}(M, \omega) \tag{1.43}$$

$$\delta_\epsilon F = \mathcal{L}_{V_\epsilon} F = \{F, H_{V_\epsilon}\} \tag{1.44}$$

The Hamiltonians can be encoded in terms of *moment map* $\mu : M \to \mathfrak{g}^*$ so that

$$H_{V_\epsilon} = \langle \epsilon, \mu \rangle. \tag{1.45}$$

with $\langle \cdot, \cdot \rangle$ being the canonical paring between $\mathfrak{g}$ and $\mathfrak{g}^*$. The dual Lie algebra $\mathfrak{g}^*$, is equipped with the canonical action G in the form of coadjoint action $Ad^*_{g^{-1}}$, so we can define *equivariant moment map* obeying

$$\mu(\Phi_g(x)) = \mathrm{Ad}^*_{g^{-1}} \mu(x), \quad \forall g \in G \tag{1.46}$$

The action of a Lie group G on $(\mathcal{M}, \omega)$ is called *Hamiltonian action* if there exists an equivariant map.

Example: Let us consider $\mathcal{M} = \mathbb{R}^2$ with coordinates p, x and symplectic form $\omega = dp \wedge dx$. The infinitesimal rotations

$$\delta_\alpha x = -\alpha p, \quad \delta_\alpha p = \alpha x \tag{1.47}$$

are generated by the Hamiltonian vector field

$$V_\alpha = \alpha x \frac{\partial}{\partial p} - \alpha p \frac{\partial}{\partial x}, \quad \iota_{V_\alpha} \omega = dH_\alpha \tag{1.48}$$

with Hamiltonian

$$H_\alpha = \frac{1}{2} \alpha (p^2 + x^2). \tag{1.49}$$

1.8 Quantization

The quantum description of the point particle consists of the Hilbert space of states $\mathcal{H}$ and the time evolution operator, quantum Hamiltonian $\hat{H}$. The wavefunction of the system Ψ

obeys the Schrodinger equation

$$-i\hbar\partial_t\Psi = \hat{H}\Psi \tag{1.50}$$

Let us review the construction for the Hilbert space and quantum Hamiltonian.

Definition: A *quantization* of the symplectic manifold (M, ω) provides

- A *Hilbert space* $\mathcal{H}$.
- A map assigning a self-adjoint operator, *quantum observable*, $\hat{f}$ to every function *classical observable* $f \in C^\infty(M)$ satisfying:

  1. Linearity
  $$\widehat{af + bg} = a\hat{f} + b\hat{g}, \;\; \forall f, g \in C^\infty(M). \tag{1.51}$$
  2. Constants map to constant operators
  $$\hat{1} = 1_{\mathcal{H}}. \tag{1.52}$$
  3. Hermiticity
  $$\hat{f}^\dagger = \hat{f}, \;\; \forall f \in C^\infty(M). \tag{1.53}$$
  4. Dirac's correspondence principle
  $$[\hat{f}, \hat{g}] = -i\hbar\widehat{\{f, g\}}. \tag{1.54}$$
  5. $\mathcal{H}$ is an irreducible representation of any irreducible subalgebra of $C^\infty(M)$. If $\{g_1, ..., g_n\}$ is a complete set of observables, $\{\hat{g}_1, ..., \hat{g}_n\}$ is a complete set of operators. Being the complete set of observables is such that
  $$\forall k: \; \{f, g_k\} = 0 \;\; \Rightarrow f \in \mathbb{C}. \tag{1.55}$$
  Similarly the complete set of operators $\{\hat{g}_k\}$ implies that any operator $\hat{A}$ commuting with them is scalar i.e.
  $$\forall k \;\; [\hat{A}, \hat{g}_k] = 0 \;\; \Rightarrow \hat{A} = \lambda \cdot 1_{\mathcal{H}}, \;\; \lambda \in \mathbb{C}. \tag{1.56}$$

Proposition: Requirements (axioms) are too strong so we need to relax some of them to get quantization. There is an extensive discussion of this proposition in the literature.

There are different types of quantization, depending on the assumptions that we preserve:

- **Deformation (Canonical) quantization**: Dirac's correspondence principle is required only asymptotically as $\hbar \to 0$ - deformation of the usual product in $C^\infty(M)$.
- **Geometric quantization**: Consider irreducible representations of only some irreducible subalgebras of $C^\infty(M)$.
- **Path integral quantization**.

1.9 Canonical quantization

The canonical quantization is typically the first quantization that we encounter in a physics courses. The idea is simple enough: Given trivial symplectic manifold $\mathbb{R}^2$ with canonical symplectic structure

$$\omega = dp \wedge dq \tag{1.57}$$

or equivalently a canonical Poisson bracket

$$\{p, q\} = 1 \tag{1.58}$$

we represent

$$q \to \hat{q} = q, \;\; p \to \hat{p} = -i\hbar \frac{\partial}{\partial q}. \tag{1.59}$$

The Hilbert space is

$$\mathcal{H} = L^2(\mathbb{R}), \;\; \langle \psi, \eta \rangle = \int_{\mathbb{R}} \bar{\psi}(q)\eta(q)\, dq \tag{1.60}$$

while the classical observables $f(p, q) \in C^\infty$ are represented

$$f(p, q) \mapsto \hat{f} =: f(\hat{p}, \hat{q}) : \tag{1.61}$$

with : : stands for ordering choice. Let us chose the so called *normal ordering* with all derivatives being to the right.

Operators $\hat{p}$ and $\hat{q}$ are hermitian operators. Indeed simple check shows that

$$\begin{aligned}\langle \hat{q}^\dagger \psi, \eta\rangle =&\langle \psi, \hat{q}\eta\rangle = \int_{\mathbb{R}} \bar{\psi}(q)\cdot \hat{q}\eta(q)\, dq = \int_{\mathbb{R}} \bar{\psi}(q)\cdot q\eta(q)\, dq \\ &= \int_{\mathbb{R}} q\bar{\psi}(q)\cdot \eta(q)\, dq = \int_{\mathbb{R}} \overline{q\psi(q)}\cdot \eta(q)\, dq = \int_{\mathbb{R}} \overline{\hat{q}\psi}(q)\cdot \eta(q)\, dq \\ &= \langle \hat{q}\psi, \eta\rangle\end{aligned} \tag{1.62}$$

and

$$\begin{aligned}\langle \hat{p}^\dagger \psi, \eta\rangle &= \langle \psi, \hat{p}\eta\rangle = \int_{\mathbb{R}} \bar{\psi}(q)\cdot \hat{p}\eta(q)\, dq = -i\hbar \int_{\mathbb{R}} \bar{\psi}(q)\partial_q \eta(q)\, dq \\ &= i\hbar \int_{\mathbb{R}} \partial_q \bar{\psi}(q)\cdot \eta(q)\, dq = \int_{\mathbb{R}} \overline{-i\partial_q \psi(q)}\cdot \eta(q)\, dq = \int_{\mathbb{R}} \overline{\hat{p}\psi(q)}\cdot \eta(q)\, dq \\ &= \langle \hat{p}\psi, \eta\rangle\end{aligned} \tag{1.63}$$

Remark: In Canonical quantization the Dirac's correspondence principle only holds at $\mathcal{O}(\hbar)$ and can be corrected at higher orders!

1.10 Time evolution

The particle trajectory in phase space is a solution to Hamilton's equations

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p} \tag{1.64}$$

which we can rewrite in terms of Poisson bracket

$$\dot{p} = \{H, p\}, \quad \dot{q} = \{H, q\} \tag{1.65}$$

In a similar way we can describe the time evolution of the classical observable

$$\dot{\mathcal{O}}(p, q) = \dot{q}\frac{\partial \mathcal{O}}{\partial q} + \dot{p}\frac{\partial \mathcal{O}}{\partial p} = \frac{\partial H}{\partial p}\frac{\partial \mathcal{O}}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial \mathcal{O}}{\partial p} = \{H, \mathcal{O}(p, q)\} \tag{1.66}$$

In quantum theory we replace the Poisson bracket by commutator to arrive into

$$\dot{\hat{\mathcal{O}}} = \frac{i}{\hbar}[\hat{H}, \hat{\mathcal{O}}] \tag{1.67}$$

what can be solved

$$\hat{\mathcal{O}}(t) = e^{\frac{i}{\hbar}\hat{H}t}\hat{\mathcal{O}}(0)e^{-\frac{i}{\hbar}\hat{H}t} \tag{1.68}$$

The time dependent operators with stationary wave-functions ψ is known as the *Heisenberg picture*. There is a physically equivalent description - *Schrodinger's picture* with stationary operators and time-dependent wave functions $\psi(t)$. The observables - probability of observing certain eigenvalue for Hermitian operator are the same in both pictures

$$p(t) = \langle\psi|\mathcal{O}|\psi\rangle = \int_{\mathbb{R}} dq\ \bar{\psi}(q)\hat{\mathcal{O}}(t)\psi(q) = \int_{\mathbb{R}} dq\ \bar{\psi}(q,t)\hat{\mathcal{O}}(0)\psi(q,t) \tag{1.69}$$

if we describe the time evolution of wave-functions via unitary operator

$$\psi(t) = U\psi(0) = e^{-\frac{i}{\hbar}\hat{H}t}\psi(0) \tag{1.70}$$

Equivalently we can say that $\psi(t)$ solves the Scrodinger's equation

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi. \tag{1.71}$$

1.11 Path integral in Hamiltonian formulation

We have already observed that the classical equations of motion

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p} \tag{1.72}$$

follow from the extremum of the action

$$S[p(t), q(t)] = \int_0^T (p\dot{q} - H(p,q)). \tag{1.73}$$

An extremum of the action determines the saddle points of the partition function

$$Z(T) = \int_{Maps(I,M)} \mathcal{D}q\mathcal{D}p\ \ e^{\frac{i}{\hbar}S[p(t),q(t)]} \tag{1.74}$$

so it is natural to ask the question about the meaning of the partition function. The answer to this question is rather interesting

$$Z(T) = \langle q_F|e^{-\frac{i}{\hbar}\hat{H}T}|q_I\rangle \tag{1.75}$$

with q_i and q_f being initial and final positions of the particle

$$q_I = q(0), \quad q_F = q(T). \tag{1.76}$$

Let us sketch the derivation of the path integral for

$$H = \frac{p^2}{2m} + V(q) \tag{1.77}$$

Let us use the completness

$$I_{\mathcal{H}} = \int dp\ |p\rangle\langle p|, \quad I_{\mathcal{H}} = \int dq\ |q\rangle\langle q|, \tag{1.78}$$

of position and momentum basises

$$\hat{q}|q\rangle = q|q\rangle, \ \ \hat{p}|p\rangle = p|p\rangle \tag{1.79}$$

to rewrite

$$Z(T) = \int dp \langle q_F|e^{-\frac{i}{\hbar}\hat{H}T}|p\rangle\langle p|q_I\rangle \tag{1.80}$$

Let us split the interval $[0, T]$ into N parts and introduce notations

$$p_k = p(k\Delta t), \ \ q_k = q(k\Delta t) \tag{1.81}$$

Then we can rewtrite the partition function

$$\begin{aligned} Z(T) &= \int dp_1....dp_N \quad \int dq_1...dq_{N-1} \prod_{k=0}^{N} \langle q_k|e^{-\frac{i}{\hbar}\hat{H}\Delta t}|p_k\rangle\langle p_k|q_{k-1}\rangle \\ &= \int dp_1....dp_N \quad \int dq_1...dq_{N-1} \prod_{k=0}^{N} \frac{1}{2\pi\hbar} \exp\left[\frac{ip_k(q_k - q_{k-1})}{\hbar} - \frac{i\Delta t}{\hbar} H(p_k, q_k)\right] \\ &= \int dp_1....dp_N \quad \int dq_1...dq_{N-1} \prod_{k=0}^{N} \frac{1}{2\pi\hbar} \exp\left[\frac{i\Delta}{\hbar}(p_k\dot{q}_k - H(p_k, q_k))\right] \\ &= \int \mathcal{D}q\mathcal{D}p \ \ e^{\frac{i}{\hbar}\int_0^T (p\dot{q} - H(p,q))} = \int_{Maps(I,\mathbb{R}^2)} \mathcal{D}q\mathcal{D}p \ \ e^{\frac{i}{\hbar}S[p(t),q(t)]} \end{aligned} \tag{1.82}$$

We used

$$\langle p|q\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{ipq}{\hbar}} \tag{1.83}$$

and

$$
\begin{aligned}
\langle q|e^{-\frac{i}{\hbar}\hat{H}\Delta t}|p\rangle = \langle q|e^{-\frac{i}{\hbar}\left(\frac{\hat{p}^2}{2m}+V(\hat{q})\right)\Delta t}|p\rangle &= \langle q|\left[1-\frac{i}{\hbar}\left(\frac{\hat{p}^2}{2m}+V(\hat{q})\right)\Delta t+\mathcal{O}(\Delta t)^2\right]|p\rangle \\
&= \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{ipq}{\hbar}}\left(1-\frac{i\Delta t}{\hbar}\frac{p^2}{2m}-\frac{i\Delta t}{\hbar}V(q)+\mathcal{O}(\Delta t)^2\right) \\
&= \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{ipq}{\hbar}-\frac{i\Delta t}{\hbar}H(p,q)}+\mathcal{O}(\Delta t)^2
\end{aligned}
\tag{1.84}
$$

and

$$
e^{t(A+B)} = e^{tA}e^{tB}(1+t^2[A,B]+\mathcal{O}(t^3)) \tag{1.85}
$$

1.12 Path integral for symplectic manifold*

The Hamiltonian formulation of the path integral

$$
\langle q_F|e^{-\frac{i}{\hbar}\hat{H}T}|q_I\rangle = \int_{Maps(I,\mathbb{R}^2)}\mathcal{D}q\mathcal{D}p\ \ \exp\left(\frac{i}{\hbar}\int_0^T dt(p\dot{q}-H(p,q))\right) \tag{1.86}
$$

The boundary conditions for map space are

$$
\gamma^q(0)=q_I,\quad \gamma^q(T)=q_F,\ \ \forall\gamma\in Maps(I,\mathbb{R}^2) \tag{1.87}
$$

which geometrically means that γ stretches between two Lagrangian submanifolds

$$
\gamma(0)\in\mathcal{L}_{q_I}=\{(p,q_I)|\ p\in\mathbb{R}\},\quad \gamma(T)\in\mathcal{L}_{q_F}=\{(p,q_F)|\ p\in\mathbb{R}\} \tag{1.88}
$$

The $p\dot{q}$ term in action is the pullback of canonical 1-form $\theta = pdq$ on $\mathbb{R}^2 = T^*\mathbb{R}$

$$
\int_0^T dt\ p\dot{q} = \int_I \gamma^*\theta,\ \ d\theta=\omega,\ \ \theta\Big|_{\mathcal{L}_q}=0 \tag{1.89}
$$

On QM side the states $|q\rangle$ are quasi-classicaly supported on $\mathcal{L}_q$. We can define the path integral quantization via the formula

$$
\langle q_F|\prod\hat{\mathcal{O}}_i|q_I\rangle = \int_{Maps(I,\mathbb{R}^2)}\mathcal{D}q\mathcal{D}p\ \ \exp\left(\frac{i}{\hbar}\int_0^T dtp\dot{q}\right)\prod\mathcal{O}_i(p(t),q(t)) \tag{1.90}
$$

The trace formula

$$\mathrm{Tr}\left(\prod \hat{\mathcal{O}}_i\right) = \int_{Maps(S^1,\mathbb{R}^2)} \mathcal{D}\gamma \ \ \exp\left(\frac{i}{\hbar}\int_{S^1}\gamma^*\theta\right) \prod \mathcal{O}_i(\gamma(t_i)) \tag{1.91}$$

allows us to drop the polarization dependence.

1.13 Path integral in Lagrangian formulation

We can use the saddle point approximation to evaluate the momentum space path integral

$$Z_T = \int_{Maps(I,T^*X)} \mathcal{D}q\mathcal{D}p \ \ e^{\frac{i}{\hbar}S[p(t),q(t)]} \tag{1.92}$$

The critical points of the action

$$\frac{\delta S}{\delta p(t)} = \dot{q}(t) - \frac{\partial H}{\partial p}(t) \tag{1.93}$$

can be solved for $p(t)$ in terms of $q(t)$ and $\dot{q}(t)$. The equation above is the part of the Legendre transform from Hamiltonian $H(p,q)$ to Lagrangian $L(q,\dot{q})$. The action S, evaluated on solution to saddle point equation is the integral of Lagrangian

$$S[p_{cl}(t),q(t)] = \int (p_{cl}\dot{q} - H(p_{cl},q))\, dt = \int dt\ L(q,\dot{q}) \tag{1.94}$$

The path integral at leading order in $\hbar$ becomes an integral

$$Z_T = \langle x_F | e^{\frac{i}{\hbar}\hat{H}T} | x_I\rangle = \int_{\mathcal{T}} \mathcal{D}x \ \ e^{\frac{i}{\hbar}S[x(t)]} + \mathcal{O}(\hbar^0) \tag{1.95}$$

over over space of trajectories

$$\mathcal{T} = C^\infty Map([0,T],\mathbb{R}), \ \ x(0) = x_I, \quad x(T) = x_F \tag{1.96}$$

where

$$S : \mathcal{T} \to \mathbb{R}, \ \ S[x(t)] = \int_I L(x,\dot{x})dt = \int_I dt \ \left(\frac{1}{2}m\dot{x}^2 - V(x)\right), \tag{1.97}$$

being the action functional we discussed before. Similarly we can express more complicated expressions, amplitudes for example

$$\langle x_F | e^{\frac{i}{\hbar}\hat{H}t_1}\hat{\mathcal{O}}_1 e^{\frac{i}{\hbar}\hat{H}(t_2-t_1)}....\hat{\mathcal{O}}_n e^{\frac{i}{\hbar}\hat{H}(T-t_n)}|x_I\rangle = \int_{\mathcal{T}} \mathcal{D}x \ \ e^{\frac{i}{\hbar}S}\mathcal{O}_1(x(t_1))\cdot ...\mathcal{O}(x(t_n)) \tag{1.98}$$

We can describe the path integral formulation of the trace

$$\mathrm{Tr}\mathcal{O} = \int dx \langle x|\mathcal{O}|x\rangle \tag{1.99}$$

in the form

$$\mathrm{Tr}(e^{\frac{i}{\hbar}\hat{H}t_1}\hat{\mathcal{O}}_1 e^{\frac{i}{\hbar}\hat{H}(t_2-t_1)}....\hat{\mathcal{O}}_n e^{\frac{i}{\hbar}\hat{H}(T-t_n)}) = \int_{\mathcal{T}_0} \mathcal{D}x \;\; e^{\frac{i}{\hbar}S}\mathcal{O}_1(x(t_1)) \cdot ...\mathcal{O}(x(t_n)) \tag{1.100}$$

with $\mathcal{T}_0$ being loop space for $\mathbb{R}$

$$\mathcal{T}_0 = C^\infty Map(S^1, \mathbb{R}). \tag{1.101}$$

There are multiple advantages in representation of the QM in the form of integral

- We can use saddle point approximation method to evaluate the integral as power series expansion in $\hbar$ also known as the Feynmann diagram expansion.

$$Z(\hbar) = \int_\gamma dx \; e^{i\hbar^{-1}S(x)} = \sum_{x_0 : S(x_0)=0} e^{i\hbar^{-1}S(x) - \frac{1}{2}\log S'(x_0) + \mathcal{O}(\hbar)} \tag{1.102}$$

- We can perform change of integration variables in integral. For example we can use it to go from position space representation of the evolution operator to the momentum space.

- We can use the simplifications of the integration of a total derivative. It will be very useful in our discussion of supersymmetry.

- We can use the analytic continuation idea to describe the tunneling effects in QM which are not the saddle point expansion around the classical trajectory, due to absence of any (real) classical solutions.

Remark: We can and in part we will discuss the relation of the Morse theory and SQM without using the Path integral. However the Path integral has so many applications in modern physics and math, so getting some knowledge of it will be beneficial for everyone.

1.14 Geometric quantization*

Definition: The *quantization* of symplectic manifold (M, ω) is the prequantization $(M, \omega, L, h, \nabla)$ of M together withe choice of polarization $P \subset TM$ of M. The Hilbert space of theory $\mathcal{H}_P$

is the space of P-polarized, square integrable sections of L. The observables are the (prequantized) classical observables, compatible with the choice of polarization.

1. **Definition**: A symplectic manifold (M, ω) is *prequantizable* when there exists a line bundle $\pi : L \to M$ with Hermitian structure h and connection $\nabla : \Gamma(M, L) \to \Gamma(M, L \otimes \Omega^1(M))$, such that the curvature F_∇ is proportional to the symplectic form $\omega = i\hbar F_\nabla$.

   Theorem: Let ω be a closed 2-form on M such that $[\omega] \in H^2(M, \mathbb{R}) \subset H^2(M, \mathbb{Z})$. Then there exists a complex line bundle $L \to M$ and a connection ∇ such that $\omega = \frac{i}{2\pi} F_\nabla$. (In particular, this means that $c_1(L) = [\omega]$.)

   Corollary: The symplectic manifold (M, ω) is prequantizible if

$$c_1(L) = \frac{i}{2\pi}[F] = \left[\frac{\omega}{2\pi\hbar}\right] \in H^2(M, \mathbb{Z}). \tag{1.103}$$

   The statement above is equivalent

$$\frac{1}{2\pi\hbar}\int_\Sigma \omega \in \mathbb{Z} \tag{1.104}$$

   for any closed two cycle $\Sigma \in H_2(M, \mathbb{Z})$.

   Remark: The integrality of the periods of ω is typically discussed in QM course under the name *Bohr-Sommerfeld (BS) quantization* and typically formulated in the form

$$\int_C p_i dq^i = 2\pi\hbar(n + \epsilon_C), \tag{1.105}$$

   with C being a closed contour and ϵ_C being some quantum correction. If there is a surface S such that $\partial S = C$ then we can rewrite an expression above into

$$\frac{1}{2\pi\hbar}\int_C p_i dq^i = \frac{1}{2\pi\hbar}\int_S \omega, \tag{1.106}$$

   with $\omega = dp_i \wedge dq^i$ being the canonical form on phase space T^*X for a particle moving on configuration space X. The BS contours are chosen on the surface of constant energy

$$H(p, q) = E. \tag{1.107}$$

2. The Hilbert space $\mathcal{H}$ is the space $\Gamma_{pol}(M, L)$ of *polarized* sections

$$\Gamma_{pol}(M, L) = \{s \in \Gamma(M, L) \, | \nabla_X s = 0 \;\; \forall X \in P\} \tag{1.108}$$

Definition: A *polarization* of a symplectic manifold is a foliation of the manifold by Lagrangian subspaces. That is, a sub-bundle $P \subset TM$ such that

- P_m is closed under commutator: for all $X, Y \in P_m \subset T_m M$

$$[X, Y]\Big|_m \in P_m \tag{1.109}$$

- P_m is Lagrangian:

$$\omega|_{P_m} = 0. \tag{1.110}$$

3. The observables map is defined to be

$$\hat{f}(\sigma) = -i\hbar\nabla_{X_f}\sigma + f\sigma, \;\; f \in C^\infty(M), \;\; \sigma \in \mathcal{H}. \tag{1.111}$$

where

$$df + \iota_{X_f}\omega = 0 \tag{1.112}$$

Observables obey the Dirac's correspondence principle i.e.

$$[\hat{f}, \hat{g}] = -i\hbar\widehat{\{f, g\}}. \tag{1.113}$$

4. Observable f is compatible with polarization P i.e.

$$\nabla_X s = 0 \;\; \Rightarrow \nabla_X \hat{f} s = 0 \tag{1.114}$$

if and only if

$$[X, X_f] \in P. \tag{1.115}$$

Example: Let us apply the geometric quantization approach to the simplest example of 2d symplectic manifold $M = \mathbb{R}^2$ with canonical symplectic structure $\omega = dp \wedge dq$. The connection is

$$\nabla = d + A, \quad A = -\frac{i}{\hbar}pdq \quad i\hbar F = i\hbar dA = \omega \tag{1.116}$$

For the classical observables p and q the corresponding Hamilton vector fields

$$X_p = \partial_q, \quad X_q = -\partial_p \tag{1.117}$$

lead to the operator representation

$$\hat{p} = -i\hbar\nabla_{X_p} + p = -i\hbar\iota_{X_p}(dq\partial_q + dp\partial_p - \frac{i}{\hbar}pdq) + p = -i\hbar\partial_q, \tag{1.118}$$

and

$$\hat{q} = -i\hbar\nabla_{X_q} + q = i\hbar\partial_p + q. \tag{1.119}$$

The real polarized sections are

$$\partial_p s(p,q) = 0 \ \ \Rightarrow s(p,q) = s(q). \tag{1.120}$$

The $\hat{p}$ and $\hat{q}$ are compatible observables.