Supersymmetric Quantum Mechanics and Morse theory

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Lectures 12-15: $d = 0$ **Supersymmetry v0**

Contents

1 $d = 0$ Supersymmetry

Let us consider an integral

$$Z_W = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-W^2} W' dx \tag{1.1}$$

for $W(x)$ being polynomial of degree n i.e.

$$W(x) = x^n + a_{n-1} x^{n-1} + ... + a_0. \tag{1.2}$$

In physics literature integrals similar to (1.1) appear in many different models. The Z_W is often referred as the *partition function* while the argument of exponent - as *action of theory*.

We can perform a change of variables

$$y = W(x), \;\; dy = W'(x) dx \tag{1.3}$$

to turn the integral (1.1) into the Gaussian inegral

$$Z_W = \frac{1}{\sqrt{\pi}} \int_{W(-\infty)}^{W(+\infty)} dy \; e^{-y^2}. \tag{1.4}$$

The integration result depends on the limits of integration, which are determined by degree of polynomial n and the sign of the top coefficient a_n

$$Z_W = \begin{cases} 0 & n \bmod 2 = 0, \quad W(\pm\infty) = +\infty, \\ 1 & n \bmod 2 = 1, a_n > 0 \quad W(\pm\infty) = \pm\infty, \\ -1 & n \bmod 2 = 1, a_n < 0 \quad W(\pm\infty) = \mp\infty. \end{cases} \tag{1.5}$$

1.1 Grassmann-odd symmetry

Let us use the pair of Grassmann-odd variables ψ and $\bar{\psi}$ to lift the W' into the action

$$W'(x) = \int_{\mathbb{R}^{0|2}} d\psi d\bar{\psi} \;\; e^{W'\psi\bar{\psi}}, \tag{1.6}$$

while the partition function becomes

$$Z_W = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}^{1|2}} dx d\psi d\bar{\psi} \;\; e^{-W^2 + W'\psi\bar{\psi}} = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}^{1|2}} dx d\psi d\bar{\psi} \;\; e^{-S(x,\psi,\bar{\psi})}. \tag{1.7}$$

The new action

$$S(x,\psi,\bar{\psi}) = W^2 - W'\psi\bar{\psi} \tag{1.8}$$

depends on both even and odd variables and usually denoted as *supersymmetric action* in physics literature. The supersymmetric action is invariant under transformations

$$\begin{aligned} \delta_\epsilon x &= \bar{\epsilon}\psi + \epsilon\bar{\psi}, \\ \delta_\epsilon \psi &= 2W\epsilon, \\ \delta_\epsilon \bar{\psi} &= -2W\bar{\epsilon}. \end{aligned} \tag{1.9}$$

The parameters ϵ and $\bar{\epsilon}$ are Grassmann-odd variables i.e they obey

$$\begin{aligned} \epsilon\bar{\epsilon} = -\bar{\epsilon}\epsilon, \;\; \epsilon^2 = \bar{\epsilon}^2 = 0, \\ \{\epsilon,\psi\} = \{\epsilon,\bar{\psi}\} = \{\bar{\epsilon},\psi\} = \{\bar{\epsilon},\bar{\psi}\} = 0. \end{aligned} \tag{1.10}$$

The symmetry transfromation mixes parity even (bosonic) and parity odd (fermionic) variables and is denoted by *supersymmetry transformation* in physics literature. The change in action S

$$\begin{aligned} \delta_\epsilon S &= 2WW'\delta_\epsilon x - W''\delta_\epsilon x \cdot \psi\bar{\psi} - W'\delta_\epsilon\psi \cdot \bar{\psi} - W'\psi \cdot \delta_\epsilon\bar{\psi} \\ &= 2WW'(\bar{\epsilon}\psi + \epsilon\bar{\psi}) - W'(2W\epsilon)\bar{\psi} - W'\psi(-2\bar{\epsilon}W) = 0. \end{aligned} \tag{1.11}$$

Under the change of variables the integration measure $dxd\psi d\bar{\psi}$ transforms by super-determinant

$$dx'd\psi'd\bar{\psi}' = \text{sdet}(J)\cdot dxd\psi d\bar{\psi} \tag{1.12}$$

with J being Jacobian for the change of variables

$$\begin{aligned} x' &= x + \epsilon\bar{\psi} + \bar{\epsilon}\psi, \\ \psi' &= \psi + 2W\epsilon, \\ \bar{\psi}' &= \bar{\psi} - 2W\bar{\epsilon}. \end{aligned} \tag{1.13}$$

In matrix form the Jacobian

$$J = \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial \psi} & \frac{\partial x'}{\partial \bar{\psi}} \\ \frac{\partial \psi'}{\partial x} & \frac{\partial \psi'}{\partial \psi} & \frac{\partial \psi'}{\partial \bar{\psi}} \\ \frac{\partial \bar{\psi}'}{\partial x} & \frac{\partial \bar{\psi}'}{\partial \psi} & \frac{\partial \bar{\psi}'}{\partial \bar{\psi}} \end{pmatrix} = \begin{pmatrix} 1 & -\bar{\epsilon} & -\epsilon \\ 2W'\epsilon & 1 & 0 \\ -2W'\bar{\epsilon} & 0 & 1 \end{pmatrix}, \tag{1.14}$$

while the superdeterminant

$$\text{sdet}(J) = \text{Ber}(J) = 1 + 4W'\bar{\epsilon}\epsilon = 1 + \mathcal{O}(\epsilon,\bar{\epsilon}) \tag{1.15}$$

The superdeterminant is identity when $\epsilon = 0$ or $\bar{\epsilon} = 0$, while being corrected by the second order terms in ϵ in general case.

1.2 Localization in simple model

The supersymmetry transformations act as linear shift of ψ

$$\psi \to \psi + 2\epsilon W(x), \tag{1.16}$$

so in case $W(x)$ is non-vanishing we can use the symmetry to set ψ to zero in the action. The Grassmann integration of ψ-independent action makes the partition function vanish. In particular we can perform a change of variables

$$\begin{aligned} y &= x - \frac{\psi\bar{\psi}}{2W(x)}, \\ \chi &= \sqrt{W(x)}\psi, \\ \bar{\chi} &= \bar{\psi}, \end{aligned} \tag{1.17}$$

to make the action in action χ-independent

$$S = W^2(y) = W^2(x) - W'(x)\psi\bar{\psi}. \tag{1.18}$$

The Jacobian matrix of coordinate transformation (1.17)

$$J = \begin{pmatrix} 1 + W'\frac{\psi\bar{\psi}}{2W^2} & -\frac{\bar{\psi}}{2W(x)} & \frac{\psi}{2W(x)} \\ \frac{1}{2}\frac{W'}{\sqrt{W}}\psi & \sqrt{W} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1.19}$$

while its super-determinant

$$\begin{aligned} \mathrm{sdet}(J) &= \frac{1}{\sqrt{W}}\left[1 + W'\frac{\psi\bar{\psi}}{2W^2} - \left(-\frac{\bar{\psi}}{2W}\right)\frac{1}{\sqrt{W}}\frac{1}{2}\frac{W'}{\sqrt{W}}\psi\right] \\ &= \frac{1}{\sqrt{W}}\left[1 + W'\frac{\psi\bar{\psi}}{4W^2}\right] = \frac{1}{\sqrt{W\left[1 - W'\frac{\psi\bar{\psi}}{2W^2}\right]}} \\ &= \frac{1}{\sqrt{W(x) - W'(x)\frac{\psi\bar{\psi}}{2W(x)}}} = \frac{1}{\sqrt{W\left(x - \frac{\psi\bar{\psi}}{2W(x)}\right)}} \\ &= \frac{1}{\sqrt{W(y)}}. \end{aligned} \tag{1.20}$$

The integration measures are related via

$$dyd\chi d\bar{\chi} = \mathrm{sdet}(J)\; dxd\psi d\bar{\psi}, \;\; dxd\psi d\bar{\psi} = \sqrt{W(y)}\; dyd\chi d\bar{\chi}, \tag{1.21}$$

while the integral

$$Z_W = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}^{1|2}} dxd\psi d\bar{\psi}\;\; e^{-W^2 + W'\psi\bar{\psi}} = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}^{1|2}} dyd\chi d\bar{\chi}\;\; \sqrt{W(y)}\;\; e^{-W^2(y)} = 0 \tag{1.22}$$

vanishes because of the Grassmann integral over χ of the χ-independent expression.

In case of $W(x)$, which has zeroes we can split the integration over x into two regions: the one that does not contain zeros of W and the small regions around zeroes. The integral over former region vanishes while the integral over latter can be written using Taylor exapansion

$$\begin{aligned} Z_W &= \sum_{x_0:W(x_0)=0} \frac{1}{\sqrt{\pi}} \int dx d\psi d\bar{\psi}\ e^{-[W'(x_0)]^2(x-x_0)^2+W'(x_0)\psi\bar{\psi}+\dots} \\ &= \sum_{x_0:W(x_0)=0} \frac{1}{\sqrt{\pi}} \sqrt{\frac{\pi}{[W'(x_0)]^2}} \cdot W'(x_0) = \sum_{x_0:W(x_0)=0} \frac{W'(x_0)}{|W'(x_0)|} \\ &= \sum_{x_0:W(x_0)=0} \operatorname{sign}(W'(x_0)). \end{aligned} \tag{1.23}$$

1.3 Intersection theory

The answer to the integral Z_W is an integer number, so it very reasonable to assume that it counts something (with a sign). As we will see later in this section Z_W counts the number of intersection points between graph of $W(x)$ and x-axis with multiplicities defined from relative orientation. In order to prove this statement we need to introduce some notations from intersection theory.

Definition: Let C_1 and C_2 being two sub-manifolds inside n-dimensional smooth manifold M. Let us assume that $\dim C_1 + \dim C_2 = \dim M$ and transversality of intersection i.e.

$$TC_1 + TC_2 = TM \tag{1.24}$$

We can define the *intersection number* between C_1 and C_2 denoted as $C_1 \cdot C_2$ via

$$C_1 \cdot C_2 = \sum_{p \in C_1 \cap C_2} \epsilon(p) \tag{1.25}$$

with $\epsilon(p)$ being the orientation of the point p, induced by the relative orientation of C_1, C_2 to M.

Example: Let $W : \mathbb{R} \to \mathbb{R}$ be a (polynomial) function. We can consider graph Γ_W for W defined as

$$\Gamma_W : \mathbb{R} \to \mathbb{R}^2 : x \mapsto (x, W(x)) \tag{1.26}$$

The dimensions

$$\dim(\Gamma_W) + \dim(\Gamma_0) = 2 = \dim(\mathbb{R}) \tag{1.27}$$

The intersection of graph Γ_W and x-axis Γ_0 is transverse while the orientation ϵ_{x_0} of the individual intersection points is

$$\epsilon(x_0) = \text{sign}(W'(x_0)). \tag{1.28}$$

The intersection number

$$\Gamma_W \cdot \Gamma_0 = \sum_{x_0 \in \Gamma_W \cap \Gamma_0} \text{sign}(W'(x_0)) \in \mathbb{Z} \tag{1.29}$$

matches with the localization formula from previous section.

We can express intersection number as an integral using the Poincare duality. For closed p-dimensional submanifold C inside a compact n-dimensional manifold M we can integrate the $\omega \in H^k(M)$ over it, what gives us a linear map

$$H^k(M) \to \mathbb{R} : \omega \mapsto \int_C \omega \tag{1.30}$$

while the Poincare duality tells us that we can represent this map as

$$\int_C \omega = \int_M \eta_C \wedge \omega, \quad \eta_C \in H^{n-k}(M) \tag{1.31}$$

We denote η_C as the *Poincare dual class* to the submanifold C. The intersection number in terms of Poincare-dual forms is

$$C_1 \cdot C_2 = \int_M \eta_{C_1} \wedge \eta_{C_2} \tag{1.32}$$

Example: The Poincare dual form for the graph $\Gamma_W \subset \mathbb{R}^2_{xy}$ is

$$\eta_{\Gamma_W} = \delta(y - W(x))(dy - W'(x)dx) \in \Omega^1(\mathbb{R}^2_{xy}) \tag{1.33}$$

Indeed let us consider an arbitrary 1-form on $\mathbb{R}^2$

$$\omega = \omega_x(x,y)dx + \omega_y(x,y)dy. \tag{1.34}$$

The integral of ω over graph $\Gamma_W : \mathbb{R} \to \mathbb{R}^2 : t \mapsto (t, W(t))$ is

$$\int_{\Gamma_W} \omega = \int_{\mathbb{R}} \Gamma_W^* \omega = \int_{-\infty}^{+\infty} dt[\omega_x(t, W(t)) + \omega_y(t, W(t))W'(t)] \tag{1.35}$$

while the $\mathbb{R}^2$ integral is

$$\begin{aligned} \int_{\mathbb{R}^2} \omega \wedge \eta_{\Gamma_W} &= \int_{\mathbb{R}^2} [\omega_x(x,y)dx + \omega_y(x,y)dy] \wedge \delta(y - W(x))[dy - W'(x)dx] \\ &= \int_{\mathbb{R}^2} \delta(y - W(x))[\omega_x(x,y)dx \wedge dy + W'(x)\omega_y(x,y)dx \wedge dy] \\ &= \int_{\mathbb{R}^2} dx\ [\omega_x(x, W(x)) + W'(x)\omega_y(x, W(x))]. \end{aligned} \tag{1.36}$$

Let us describe some useful properties of Poincare dual forms.

- *Linearity*

$$\eta(\alpha C_1 + \beta C_2) = \alpha \eta_{C_1} + \beta \eta_{C_2} \tag{1.37}$$

  follows from

$$\begin{aligned} \int_{\alpha C_1 + \beta C_2} \omega = \alpha \int_{C_1} \omega + \beta \int_{C_2} \omega = \alpha \int_M \omega \wedge \eta_{C_1} + \beta \int_M \omega \wedge \eta_{C_2} \\ = \int_M \omega \wedge (\alpha \eta_{C_1} + \beta \eta_{C_2}) \end{aligned} \tag{1.38}$$

- *Boundary*

$$\eta_{\partial S} = d\eta_S \tag{1.39}$$

  follows from

$$\int_M \omega \wedge \eta_{\partial S} = \int_{\partial S} \omega = \int_S d\omega = \int_M d\omega \wedge \eta_S = \pm \int_M \omega \wedge d\eta_S \tag{1.40}$$

- *Intersection*

$$\eta_{C_1 \cap C_2} = \eta_{C_1} \wedge \eta_{C_2} \tag{1.41}$$

Proposition: The intersection number is the topological invariant, i.e that it is invariant under the continuous deformations.

Proof: We can express continuous deformation of C_1 in the form

$$C_1 \to C_1' = C_1 + \partial S \tag{1.42}$$

The corresponding intersection number in integral form

$$\begin{aligned} C_1' \cdot C_2 = \int_M \eta_{C_1'} \wedge \eta_{C_2} = \int_M (\eta_{C_1} + \eta_{\partial S}) \wedge \eta_{C_2} \\ = C_1 \cdot C_2 \pm \int_M d\eta_S \wedge \eta_{C_2} = C_1 \cdot C_2 \pm \int_M d(\eta_S \wedge \eta_{C_2}) \\ = C_1 \cdot C_2 \end{aligned} \tag{1.43}$$

Example: The intersection number between graph of Γ and x-axis in integral representation

$$\begin{aligned} \Gamma_W \cdot \Gamma_0 &= \int_{\mathbb{R}^2} \delta(y - W(x))(dy - W'(x)dx) \wedge \delta(y)dy \\ &= -\int_{\mathbb{R}^2} W'(x)\delta(y)\delta(y - W(x))dx \wedge dy = -\int_{\mathbb{R}} W'(x)\delta(-W(x))dx \\ &= \int_{\mathbb{R}} W'(x)\delta(W(x))dx = \Theta(W(x))\Big|_{-\infty}^{+\infty} = \Theta(W(+\infty)) - \Theta(W(-\infty)) \end{aligned} \tag{1.44}$$

matches with the result of Gaussian integral. We can use one of the properties of δ-function

$$\delta(W(x)) = \sum_{x_0 : W(x_0) = 0} \frac{\delta(x - x_0)}{|W'(x_0)|} \tag{1.45}$$

to rewrite the intersection number

$$\Gamma_W \cdot \Gamma_0 = \int_{\mathbb{R}} \sum_{x_0 : W(x_0) = 0} \frac{W'(x_0)}{|W'(x_0)|} \delta(x - x_0) = \sum_{x_0 : W(x_0) = 0} \frac{W'(x_0)}{|W'(x_0)|} \tag{1.46}$$

so it matches with the localization formula.

1.4 Saddle point approximation

The *saddle-point approximation* also know as the *method of steepest descent* is the approximate method to evaluate partition type of integrals

$$Z = \int_{I_x} dx \; f(x) \;\; e^{-\frac{1}{\hbar} S(x)} \tag{1.47}$$

Under the assumptions

- $f(z)$ and $S(z)$ being holomorphic functions on open, bounded, simply-connected set $\Omega_x \subset \mathbb{C}^n$, such that $I_x = \Omega_x \cap \mathbb{R}^n$ is connected
- $S(z)$ has finitely-many isolated critical points, i.e the only solutions to

$$\partial_i S(x_0) = 0, \quad x_0 \in I_x \tag{1.48}$$

  are points x_0 and there are finitely-many of them.
- The critical points of S are non-degenerate i.e

$$\det \partial_i \partial_j S(x_0) \neq 0, \quad \forall x_0 \tag{1.49}$$

We can approximate the integral

$$Z = (2\pi\hbar)^{\frac{n}{2}} \sum_{x_0 : \partial_i S(x_0)=0} \frac{1}{\sqrt{\det \partial_i \partial_j S(x_0)}} e^{-\frac{1}{\hbar} S(x_0)} (f(x_0) + \mathcal{O}(\hbar)) \tag{1.50}$$

Remark: The higher order terms can be organized into the sum over graphs.

Example: Let us rescale W by σ so we can apply the saddle point approximation

$$Z = \frac{1}{\sqrt{\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{-\frac{1}{\sigma^2} W^2} W' dx \tag{1.51}$$

An extrema of the exponent are at points x_0 such that

$$S' = \partial_x (W^2) = 2W(x_0) W'(x_0) = 0 \tag{1.52}$$

The second derivative

$$S'' = 2W'(x_0)W'(x_0) + 2W(x_0)W''(x_0) \tag{1.53}$$

Due to the W' factor in front the contribution from $\{x_0 | W'(x_0) = 0\}$ is trivial so saddle point formula

$$Z = \frac{1}{\sqrt{\pi\sigma^2}} (2\pi\sigma^2)^{\frac{1}{2}} \sum_{x_0 : W(x_0)=0} \frac{1}{\sqrt{2W'(x_0)W'(x_0)}} e^{-\frac{1}{\sigma^2} W^2(x_0)} (W'(x_0) + \mathcal{O}(\sigma)) \tag{1.54}$$

which after simplifications takes the form

$$Z = \sum_{x_0 : W(x_0) = 0} \text{sign}(W'(x_0))\,(1 + \mathcal{O}(\sigma)) \tag{1.55}$$

Let us observe that the leading two orders of saddle point approximation match with the localization formula, so we expect the subleading terms $\mathcal{O}(\sigma)$ to vanish. The cancellation of the higher order terms on the language of Feynmann diagrams is due to the -1 factors for fermionic loops.

1.5 Subleading terms*

$$Z = \frac{1}{\sqrt{\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{-\frac{1}{\sigma^2}W^2} W' dX = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dX d\psi d\bar{\psi} e^{-\frac{1}{\sigma^2}W^2 + \frac{1}{\sigma}W'\psi\bar{\psi}} \tag{1.56}$$

Let us further assume that he potential is of the form

$$W = X + X^2 \tag{1.57}$$

so the saddle point approximation for the integral is the sum of two contributions

$$Z = Z^{(0)} + Z^{(-1)}, \quad Z^{(X_0)} = \text{sign}(W'(X_0))\,(1 + \mathcal{O}(\sigma)) \tag{1.58}$$

Let us focus on the first contribution. Near the $X_0 = 0$ saddle point we can represent

$$X = X_0 + \xi \tag{1.59}$$

so the action becomes

$$S = W^2(X) - \sigma W'(X)\psi\bar{\psi} = \xi^2 + 2\xi^3 + \xi^4 - \sigma(1 + 2\xi)\psi\bar{\psi} \tag{1.60}$$

Let us do the change of variables

$$\xi = \sigma x \tag{1.61}$$

so the integral

$$Z^{(0)} = \frac{1}{\sqrt{\pi}} \int_{-\epsilon}^{\epsilon} d\xi d\psi d\bar{\psi} \;\; e^{-\frac{1}{\sigma^2}S(\xi,\psi,\bar{\psi})} = \frac{1}{\sqrt{\pi}} \int_{-\frac{\epsilon}{\sigma}}^{\frac{\epsilon}{\sigma}} \sigma dx d\psi d\bar{\psi} \;\; e^{-\frac{1}{\sigma^2}S(\sigma x,\psi,\bar{\psi})} \tag{1.62}$$

In the limit $\sigma \ll 1$ we can replace the integration region $[-\frac{\epsilon}{\sigma}, \frac{\epsilon}{\sigma}]$ by the whole real line $\mathbb{R}$ so that

$$Z^{(0)} = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}^{1|2}} dx d\psi d\bar{\psi} \;\; e^{-(x^2+2\sigma x^3+\sigma^2 x^4-(1+2x\sigma)\psi\bar{\psi})} = \langle e^{-2\sigma x^3-\sigma^2 x^4+2x\sigma\psi\bar{\psi}} \rangle \tag{1.63}$$

where we introduced notation

$$\langle F(x,\psi,\bar{\psi}) \rangle = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}^{1|2}} dx d\psi d\bar{\psi} \;\; e^{-x^2+\psi\bar{\psi}} F(x,\psi,\bar{\psi}) \tag{1.64}$$

with normalization chosen so that

$$1 = \langle 1 \rangle = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}^{1|2}} dx d\psi d\bar{\psi} \;\; e^{-x^2+\psi\bar{\psi}} \tag{1.65}$$

Let us expand the exponent

$$\begin{aligned} Z^{(0)} &= \langle e^{-2\sigma x^3-\sigma^2 x^4+2x\sigma\psi\bar{\psi}} \rangle \\ &= \langle 1 - 2\sigma x^3 - \sigma^2 x^4 + 2x\sigma\psi\bar{\psi} + \frac{1}{2}(-2\sigma x^3 - \sigma^2 x^4 + 2x\sigma\psi\bar{\psi})^2 + ... \rangle \\ &= \langle 1 - 2\sigma x^3 + 2x\sigma\psi\bar{\psi} - \sigma^2 x^4 + 2\sigma^2 x^6 - 4\sigma^2 x^4 \psi\bar{\psi} \rangle + \mathcal{O}(\sigma^3) \\ &= 1 - 2\sigma\langle x^3 \rangle + 2\sigma \langle x\psi\bar{\psi} \rangle - \sigma^2 \langle x^4 \rangle + 2\sigma^2 \langle x^6 \rangle - 4\sigma^2 \langle x^4 \psi\bar{\psi} \rangle + \mathcal{O}(\sigma^3) \end{aligned} \tag{1.66}$$

The Gaussian integrals are even so

$$\langle x^{2n+1} \rangle = \langle x^{2n+1} \psi\bar{\psi} \rangle = 0 \tag{1.67}$$

while the even powers can be evaluated using

$$\begin{aligned} \langle x^{2k} (\psi\bar{\psi})^m \rangle &= \left(\frac{\partial}{\partial b}\right)^m \left(-\frac{\partial}{\partial a}\right)^k \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}^{1|2}} dx d\psi d\bar{\psi} \;\; e^{-ax^2+b\psi\bar{\psi}} \Big|_{a=b=1} \\ &= \left(\frac{\partial}{\partial b}\right)^m \left(-\frac{\partial}{\partial a}\right)^k \frac{b}{\sqrt{a}} \Big|_{a=b=1} \end{aligned} \tag{1.68}$$

The leading order correction becomes

$$\begin{aligned} Z^{(0)} &= 1 - \sigma^2 \langle x^4 \rangle + 2\sigma^2 \langle x^6 \rangle - 4\sigma^2 \langle x^4 \psi\bar{\psi} \rangle + \mathcal{O}(\sigma^3) \\ &= 1 - \sigma^2 \frac{3}{4} + 2\sigma^2 \frac{15}{8} - 4\sigma^2 \frac{3}{4} + \mathcal{O}(\sigma^3) \\ &= 1 + \mathcal{O}(\sigma^3). \end{aligned} \tag{1.69}$$

1.6 Deformation of distribution

We can use the integral representation for δ-function

$$\delta(x) = \frac{1}{2\pi} \int dp \; e^{ipx} \tag{1.70}$$

to rewrite

$$\Gamma_W \cdot \Gamma_0 = \int_{\mathbb{R}} W'(x)\delta(W(x))dx = \frac{1}{2\pi} \int dxdpd\psi d\bar{\psi} \;\; e^{ipW(x)+\psi\bar{\psi}W'(x)}. \tag{1.71}$$

The action is invariant under the supersymmetry transformations

$$\begin{aligned} \delta_\epsilon x &= \bar{\epsilon}\psi + \epsilon\bar{\psi}, \\ \delta_\epsilon \psi &= -ip\epsilon, \\ \delta_\epsilon \bar{\psi} &= ip\bar{\epsilon} \\ \delta_\epsilon p &= 0. \end{aligned} \tag{1.72}$$

It is useful to introduce generators of supersymmetries in the form of vector fields Q and $\bar{Q}$ such that

$$\delta_\epsilon F(x, \psi, \bar{\psi}, p) = \epsilon Q(F) + \bar{\epsilon}\bar{Q}(F) \tag{1.73}$$

for arbitrary function F. The generators take the form

$$Q = \bar{\psi}\frac{\partial}{\partial x} - ip\frac{\partial}{\partial \psi}, \;\; \bar{Q} = \psi\frac{\partial}{\partial x} + ip\frac{\partial}{\partial \bar{\psi}} \tag{1.74}$$

We can represent the smeared version of δ-function in the integral form

$$\delta(x) = \frac{1}{\sqrt{\pi}} \lim_{\sigma\to 0} \frac{1}{\sigma} e^{-\frac{W(x)^2}{\sigma^2}} = \frac{1}{2\pi} \lim_{\sigma \to 0} \int dp \; e^{ipW(x)-\frac{1}{4}\sigma^2 p^2} \tag{1.75}$$

The two integral representations differ by Q-exact term

$$-\frac{1}{4}\sigma^2 p^2 = Q\left(-\frac{i}{4}\sigma^2 \psi p\right) \tag{1.76}$$

Different values of σ describe three different situations

- Gaussian integral, being toy model of quantum system

$$Z_W^{\sigma=1} = Z_W = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-W^2} W' dx = \Theta(W(+\infty)) - \Theta(W(-\infty)) \tag{1.77}$$

- Geometric description

$$Z_W^{\sigma=0} = \Gamma_W \cdot \Gamma_0 = \int_{\mathbb{R}} W'(x)\delta(W(x))dx = \sum_{x_0 : W(x_0)=0} \text{sign}(W'(x_0)) \tag{1.78}$$

- Classical limit, saddle point approximation to integral

$$Z_W^{\sigma \ll 1} = \frac{1}{\sqrt{\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{-\frac{1}{\sigma^2}W^2} W' dx = \sum_{x_0 : W(x_0)=0} \text{sign}(W'(x_0)) \left(1 + \mathcal{O}(\sigma)\right) \tag{1.79}$$

1.7 Deformation of potential

The partition function SUSY is invariant under continuous deformations. Let us consider $f(x)$ with finite support on $\mathbb{R}$. The geometric interpretation is that such f describes the homological deformation of the graph Γ_W i.e.

$$\Gamma_W \to \Gamma_{\tilde{W}} = \Gamma_W + \partial\Sigma_f \tag{1.80}$$

with Σ_f being surface

$$\Sigma_f = \{(x, y) \in \mathbb{R}^2 | W(x) \leq y \leq W(x) + f(x)\} \tag{1.81}$$

The partition function for deformed potential

$$Z_{\tilde{W}} = \frac{1}{2\pi} \int dx dp d\psi d\bar{\psi} \;\; e^{ip\tilde{W}(x) + \psi\bar{\psi}\tilde{W}'(x)} \tag{1.82}$$

We can observe that the deformed action can be expressed via

$$\begin{aligned} \tilde{S} &= ip\tilde{W}(x) + \psi\bar{\psi}\tilde{W}'(x) = ipW(x) + \psi\bar{\psi}W'(x) - \bar{\psi}\psi f'(x) + ipf(x) \\ &= S + Q(-f\psi) \end{aligned} \tag{1.83}$$

while the difference between partition functions

$$\begin{aligned}
Z_{\tilde{W}} - Z_W &= \frac{1}{2\pi}\int dxdpd\psi d\bar{\psi} \ \ \left(e^{S+Q(-f\psi)} - e^S\right) = \frac{1}{2\pi}\int dxdpd\psi d\bar{\psi} \ \ e^S\left(e^{Q(-f\psi)} - 1\right) \\
&= \frac{1}{2\pi}\int dxdpd\psi d\bar{\psi} \ \ e^S\left(-Q(f\psi) + \frac{1}{2}[Q(f\psi)]^2 + ...\right) \\
&= \frac{1}{2\pi}\int dxdpd\psi d\bar{\psi} \ \ e^S Q\left(-f\psi + \frac{1}{2}f\psi\, Q(f\psi) + ...\right) \\
&= -\frac{1}{2\pi}\int dxdpd\psi d\bar{\psi} \ \ Q\left[f\psi e^S\left(1 - \frac{1}{2}Q(f\psi) + ...\right)\right] \\
&= -\frac{1}{2\pi}\int_{\mathbb{R}^{2|2}} dxdpd\psi d\bar{\psi}\left(\bar{\psi}\frac{\partial}{\partial x} - ip\frac{\partial}{\partial \psi}\right)\left[f\psi e^S\left(1 - \frac{1}{2}Q(f\psi) + ...\right)\right] \\
&= -\frac{1}{2\pi}\int_{\mathbb{R}^{2|2}} dxdpd\psi d\bar{\psi} \ \ \bar{\psi}\frac{\partial}{\partial x}\left[f\psi e^S\left(1 - \frac{1}{2}Q(f\psi) + ...\right)\right] = 0
\end{aligned} \tag{1.84}$$

The last integral is purely boundary term in x-direction. Earlier, we assumed that $f(x)$ has compact support so the boundary contribution vanishes.

1.8 Localization via deformation

Let us formulate general localization construction for supersymmetric theories. We want to evaluate the supersymmetric partition function

$$Z = \int_M d\mu \ \ e^{-S(x,\psi)}. \tag{1.85}$$

Supersymmetric partition function imply

- Existence of Grassmann-odd symmetry, generated by vector field Q so that

$$\delta_\epsilon F(x,\psi) = \epsilon Q(F), \ \ \forall F \in C^\infty(M) \tag{1.86}$$

- The symmetry is nilpotent i.e.

$$Q^2 = \frac{1}{2}\{Q, Q\} \tag{1.87}$$

- The action $S(x,\psi)$ is invariant under the symmetry

$$\delta_e S = \epsilon Q(S) = 0 \tag{1.88}$$

- The integration measure $d\mu$ is such that the integral of Q-exact terms is trivial

$$\int_M d\mu \;\; Q(V) = 0. \tag{1.89}$$

Let us define the deformed partition function

$$Z(t) = \int_M d\mu \;\; e^{-S(x,\psi)-tQ(V)} \tag{1.90}$$

for some Grassmann-odd function V.

Proposition: The deformed partition $Z(t)$ is independent on t.

Proof: Let us consider t-derivative

$$\begin{aligned}\partial_t Z(t) = \partial_t \int d\mu \;\; e^{-S(x,\psi)-tQ(V)} = -\int d\mu \;\; Q(V) e^{-S(x,\psi)-tQ(V)} \\ = -\int d\mu \;\; Q(V e^{-S(x,\psi)-tQ(V)}) = 0\end{aligned} \tag{1.91}$$

Corollary: The deformed partition function for $t = 0$ matches with the partition function Z. We can take $t \to \infty$ limit so the integral is dominated by the critical points of $tQ(V)$

$$Z = Z(0) = Z(t) = \lim_{t\to\infty} Z(t) \tag{1.92}$$

1.9 Supersymmetry algebra

In our discussion of supersymmetric examples let us point out an important feature about the supersymmetry transformations. The action

$$S(p, x, \psi, \bar{\psi}) = ipW(x) + \psi\bar{\psi}W'(x) \tag{1.93}$$

is invariant under the supersymmetry generated by

$$Q = \bar{\psi}\frac{\partial}{\partial x} - ip\frac{\partial}{\partial \psi}, \quad \bar{Q} = \psi\frac{\partial}{\partial x} + ip\frac{\partial}{\partial \bar{\psi}} \tag{1.94}$$

with algebra

$$\{Q, \bar{Q}\} = \{Q, Q\} = \{\bar{Q}, \bar{Q}\} = 0. \tag{1.95}$$

The algebra above in physics terminology is denoted as *off-shell* $d = 0$ $N = 2$ *SUSY algebra*. Let us explain the terminology

- $d = 0$ refers to the dimension of world-sheet Σ, which is zero dimensional in our case. In quantum field theory we often study maps $\Sigma \to M$ and supersymmetry acts on the space of maps. The simplest case is when world-sheet is the single point $\Sigma = pt$ then the space of maps $Map(\Sigma, M) = M$ is the same as the target manifold M.

- $N = 2$ refers to the number of supersymmetry generators, equivalently to the dimension of supersymmetry algebra. In our case we have two generators Q and $\bar{Q}$, equivalently we have 2-dimensional algebra of supersymmetries $\mathbb{R}^{0|2}$.

- *off-shell* refers to the fact that the supersymmetry algebra is independent of the action S. The other possibility - *on-shell* supersymmetry algebra we will consider below.

We can deform action by the Q-exact term and perform Gaussian integral over p

$$e^{-S_e(x,\psi,\bar{\psi})} = \int dp\; e^{S+Q\left(-\frac{i}{4}\psi p\right)} \tag{1.96}$$

to obtain another, referred as *effective* in physics literature, action

$$S_e(x,\psi,\bar{\psi}) = W^2(x) - W'(x)\psi\bar{\psi} \tag{1.97}$$

The action S_e is invariant under the supersymmetry transformations

$$Q_e = \bar{\psi}\frac{\partial}{\partial x} + 2W(x)\frac{\partial}{\partial \psi}, \quad \bar{Q}_e = \psi\frac{\partial}{\partial x} - 2W(x)\frac{\partial}{\partial \bar{\psi}} \tag{1.98}$$

which obey algebra

$$\begin{aligned} \{Q_e, Q_e\} &= 4W'(x)\bar{\psi}\frac{\partial}{\partial \psi}, \\ \{\bar{Q}_e, \bar{Q}_\epsilon\} &= -4W'(x)\psi\frac{\partial}{\partial \bar{\psi}}, \\ \{\bar{Q}_e, Q_e\} &= 2W'(x)\left(\psi\frac{\partial}{\partial \psi} - \bar{\psi}\frac{\partial}{\partial \bar{\psi}}\right) \end{aligned} \tag{1.99}$$

The supersymmetry algebra above in the physics notations is *on-shell* $d = 0$ $N = 2$ *supersymmetry algebra*. The term *on-shell* indicates that the nontrivial commutators of supercharges are proportional to the equations of motions for effective action S_e

$$\frac{\partial S_e}{\partial \psi} = -W'(x)\bar{\psi}, \quad \frac{\partial S_e}{\partial \bar{\psi}} = W'(x)\psi \tag{1.100}$$

Therefore, as long as equations of motions are satisfied, the on-shell supersymmetry algebra is the same as the off-shell one. In previous section we observed that the saddle point approximation for partition function is a sum over critical points of S, which are identical to the solutions to the equations of motion. So we can use the localization methods for on-shell supersymmetry in the vicinity of critical point to show that the higher order terms in $\hbar$ vanish and the partition function is 1-loop exact.

In our simple example we can turn on-shell $d = 0$ $N = 2$ supersymmetry into the off-shell one by adding additional variable p. Unfortunately such method is not always possible, especially in higher dimensions. Fortunately there is a way to construct manifestly off-shell supersymmetric actions, known as the *Superspace formalism*.

1.10 Superspace formalism

Let us consider 2d Grassmann space $\mathbb{R}^{0|2}$ with coordinates θ and $\bar{\theta}$. Let us consider maps

$$\mathbb{R}^{0|2} \to \mathbb{R} \ : \ (\theta, \bar{\theta}) \mapsto x(\theta, \bar{\theta}) \tag{1.101}$$

Each map is identical to the function $\hat{x}(\theta, \bar{\theta})$, which is finite polynomial so

$$Map(\mathbb{R}^{0|2}, \mathbb{R}) = \mathbb{R}^{2|2} \tag{1.102}$$

Let us the notation x, F for even coordinates and $\psi, \bar{\psi}$ for odd coordinates on $\mathbb{R}^{2|2}$, then we can write the function

$$\hat{x}(\theta, \bar{\theta}) = x + \theta\bar{\psi} + \bar{\theta}\psi + F\theta\bar{\theta} \tag{1.103}$$

The function $\hat{x}(\theta, \bar{\theta})$ in physics literature is known as the *superfield* $x(\theta, \bar{\theta})$. In our discussion of differential forms we observed that the diffepmorphisms on M act on the functions $C^\infty(M)$ in the form of pullback map.

$$\phi^* : C^\infty(M) \to C^\infty(M) : f \mapsto \phi^*(f) = f \circ \phi, \ \ \forall \phi \in Diff(M) \tag{1.104}$$

Among diffeomorphisms of $\mathbb{R}^{0|2}$ we have translations

$$\theta \to \theta + \epsilon, \quad \bar{\theta} \to \bar{\theta} + \bar{\epsilon} \tag{1.105}$$

generated by the vector fields

$$\mathfrak{Q} = \frac{\partial}{\partial \theta}, \quad \bar{\mathfrak{Q}} = \frac{\partial}{\partial \bar{\theta}}. \tag{1.106}$$

i.e the infinitesimal translation

$$\delta_\epsilon F(\theta, \bar{\theta}) = (\epsilon \mathfrak{Q} + \bar{\epsilon} \bar{\mathfrak{Q}}) F(\theta, \bar{\theta}) \tag{1.107}$$

The group of translations on $\mathbb{R}^{0|2}$ is abelian so corresponding the algebra has trivial brackets

$$\{\mathfrak{Q}, \bar{\mathfrak{Q}}\} = \{\mathfrak{Q}, \mathfrak{Q}\} = \{\bar{\mathfrak{Q}}, \bar{\mathfrak{Q}}\} = 0. \tag{1.108}$$

which we can recognize as the off-shell $d = 0$ $N = 2$ supersymmetry algebra. The (pullback) action of the SUSY algebra on the the space of maps

$$\begin{aligned} \delta_\epsilon \hat{x}(\theta, \bar{\theta}) &= \hat{x}(\theta + \epsilon, \bar{\theta} + \bar{\epsilon}) - \hat{x}(\theta, \bar{\theta}) \\ &= (\epsilon \mathfrak{Q} + \bar{\epsilon} \bar{\mathfrak{Q}}) \hat{x}(\theta, \bar{\theta}) = \epsilon \bar{\psi} + \bar{\epsilon} \psi + \theta \bar{\epsilon} F - \bar{\theta} \epsilon F \\ &= (\epsilon Q + \bar{\epsilon} Q) \hat{x}(\theta, \bar{\theta}) = \delta_\epsilon x + \theta \delta_\epsilon \bar{\psi} + \bar{\theta} \delta_\epsilon \psi + \theta \bar{\theta} \delta_\epsilon F \end{aligned} \tag{1.109}$$

The action in components

$$\begin{aligned} \delta_\epsilon x &= (\epsilon Q + \bar{\epsilon} Q) x = \epsilon \bar{\psi} + \bar{\epsilon} \psi \\ \delta_\epsilon \psi &= (\epsilon Q + \bar{\epsilon} Q) \psi = -\epsilon F \\ \delta_\epsilon \bar{\psi} &= (\epsilon Q + \bar{\epsilon} Q) \bar{\psi} = \bar{\epsilon} F \\ \delta_\epsilon F &= (\epsilon Q + \bar{\epsilon} Q) F = 0 \end{aligned} \tag{1.110}$$

is the familiar $d = 0$ $N = 2$ SUSY algebra action on $\mathbb{R}^{2|2}$. There are additional types of superfields in our model

$$\hat{\psi}(\theta, \bar{\theta}) = D\hat{x} = \partial_\theta \hat{x} = \psi + \bar{\theta} F \tag{1.111}$$

$$\hat{\bar{\psi}}(\theta, \bar{\theta}) = \bar{D}\hat{x} = \partial_{\bar{\theta}} \hat{x} = \bar{\psi} - \theta F \tag{1.112}$$

which often reffed as *derivative superfields* or *fermionic fuperfields*. The term "fermionic" is due to the θ-independents component of superfields being Grassmann-odd, in contrast with superfield $\hat{x}(\theta, \bar{\theta})$ with constant component being Grassmann-even. The superfields $\hat{\psi}$ and $\hat{\bar{\psi}}$

also form a representations of $d = 0$ $N = 2$ SUSY algebra

$$\begin{aligned}(\epsilon Q + \bar{\epsilon} Q)\hat{\psi}(\theta, \bar{\theta}) &= D(\epsilon Q + \bar{\epsilon} Q)\hat{x} = D(\epsilon \mathfrak{Q} + \bar{\epsilon}\bar{\mathfrak{Q}})\hat{x}(\theta, \bar{\theta}) \\ &= (\epsilon \mathfrak{Q} + \bar{\epsilon}\bar{\mathfrak{Q}}) D\hat{x}(\theta, \bar{\theta}) = (\epsilon \mathfrak{Q} + \bar{\epsilon}\bar{\mathfrak{Q}})\hat{\psi}(\theta, \bar{\theta})\end{aligned} \tag{1.113}$$

We can construct SUSY invariant functions on $\mathbb{R}^{2|2}$ using integrals over $\mathbb{R}^{0|2}$ of the arbitrary functions of superfields

$$S(x, F, \psi, \bar{\psi}) = \int d\theta d\bar{\theta} \;\; \mathcal{F}(\hat{x}(\theta, \bar{\theta}), \hat{\psi}(\theta, \bar{\theta}), \hat{\bar{\psi}}(\theta, \bar{\theta})) \tag{1.114}$$

The SUSY variation

$$\begin{aligned}\delta_\epsilon S = (\epsilon Q + \bar{\epsilon}\bar{Q})S &= (\epsilon Q + \bar{\epsilon}\bar{Q}) \int d\theta d\bar{\theta} \;\; \mathcal{F}(\hat{x}(\theta, \bar{\theta}), \hat{\psi}(\theta, \bar{\theta}), \hat{\bar{\psi}}(\theta, \bar{\theta})) \\ &= \int d\theta d\bar{\theta} \;\; (\epsilon Q + \bar{\epsilon}\bar{Q})\mathcal{F}(\hat{x}(\theta, \bar{\theta}), \partial_\theta \hat{x}, \partial_{\bar{\theta}} \hat{x}) \\ &= \int d\theta d\bar{\theta} \;\; (\epsilon \mathfrak{Q} + \bar{\epsilon}\bar{\mathfrak{Q}})\mathcal{F}(\hat{x}(\theta, \bar{\theta}), \partial_\theta \hat{x}, \partial_{\bar{\theta}} \hat{x}) \\ &= \int d\theta d\bar{\theta} \;\; (\epsilon \partial_\theta + \bar{\epsilon}\partial_{\bar{\theta}})\mathcal{F}(\hat{x}(\theta, \bar{\theta}), \partial_\theta \hat{x}, \partial_{\bar{\theta}} \hat{x}) = 0\end{aligned} \tag{1.115}$$

Example: The superspace integral

$$\begin{aligned}S &= \int d\theta d\bar{\theta} \;\; [H(\hat{x}(\theta, \bar{\theta})) + \frac{1}{4}\sigma^2 D\hat{x}\bar{D}\hat{x}] \\ &= FH'(x) + \psi\bar{\psi}H''(x) + \frac{1}{4}\sigma^2 F^2\end{aligned} \tag{1.116}$$

after redefinition

$$H'(x) = W(x), \quad F = ip \tag{1.117}$$

matches with the action for supersymmetric theory we discussed before.