Supersymmetric Quantum Mechanics and Morse theory

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Lectures 9,10: De Rham cohomology.

Contents

1 de Rham cohomology

1.1 de Rham complex

Definition: For a smooth manifold M of dimension n we can define a *de Rham complex* $(\Omega^*(M), d)$

$$\Omega^0(M) \xrightarrow{d} \ldots. \xrightarrow{d} \Omega^k(\mathbb{R}) \xrightarrow{d} \Omega^{k+1}(\mathbb{R}) \xrightarrow{d} \ldots \longrightarrow \Omega^n(M) \xrightarrow{d} 0 \tag{1.1}$$

and its cohomology

$$H^p_{dR}(M) = \frac{\ker d \cap \Omega^p(M)}{\mathrm{Im} d \cap \Omega^{p-1}(M)} \tag{1.2}$$

The dimension of $H^k(M)$ is denoted by *Betti number* $b_k(M)$.

Example: The de Rham complex for the real line $\mathbb{R}$ is

$$0 \longrightarrow \Omega^0(\mathbb{R}) \xrightarrow{d} \Omega^1(\mathbb{R}) \longrightarrow 0 \tag{1.3}$$

$$f(x) \longrightarrow f'(x)dx$$

Closed 0-forms are constants and cannot be exact so

$$H^0(\mathbb{R}) = \ker d = \mathbb{R} \tag{1.4}$$

The 1-forms are all closed and all are exact since any one form

$$f dx = dF, \quad F(x) = \int_0^x f(y) dy \tag{1.5}$$

so we conclude that

$$H^1(\mathbb{R}) = 0. \tag{1.6}$$

Example: The de Rham complex for circle S^1 is

$$0 \longrightarrow \Omega^0(S^1) \longrightarrow \Omega^1(S^1) \longrightarrow 0 \tag{1.7}$$

$$f \longrightarrow f' dx \qquad .$$

Closed 0-forms are constants and cannot be exact so

$$H^0(S^1) = \ker d = \mathbb{R}. \tag{1.8}$$

The 1-forms are all closed and we can try to find a pre-image

$$f dx = dF, \quad F(x) = \int_0^x f(y) dy, \tag{1.9}$$

but the function $F(x)$ may not be a periodic function for periodic f

$$f(x + 2\pi) = f(x) \tag{1.10}$$

Indeed any periodic function can be represented in the form of Fourier series

$$f(x) = \sum_{n \in \mathbb{Z}} f_n e^{inx}, \quad f_{-n} = f_n^* \tag{1.11}$$

The $n \neq 0$ components have periodic $F(x)$ while the constant, $n = 0$ component, is not periodic

$$F(x) = \int_0^x f_0 dy = f_0 x \neq f_0(x + 2\pi) = F(x + 2\pi) \tag{1.12}$$

Thus we conclude that

$$H^1(\mathbb{R}) = \mathbb{R}. \tag{1.13}$$

Remark: Let us observe that the de Rham cohomology match the simplicial cohomology for $\mathbb{R}$ and S^1. It is generally true that de Rham cohomology match with the simplicial ones

$$H^k_{dR}(M) = H^k_{\Delta}(M). \tag{1.14}$$

1.2 Differential forms and smooth chains

We can integrate differential p-forms on M over p-dimensional chains, so we can define a paring

$$\omega(c) = \langle c, \omega \rangle = \int_c \omega \in \mathbb{R} \tag{1.15}$$

The paring allows us to describe the dual of de Rham complex which is the smooth version of the simplicial complex. The dual differential is defined via

$$\partial_k : C^\infty_k \to C^\infty_{k-1}, \quad (d_{k+1}\omega)(c) = \omega(\partial_k c) \tag{1.16}$$

We can use the Stoke's theorem for differential form integration

$$(d\omega)(c) = \int_c d\omega = \int_{\partial c} \omega = \omega(\partial c). \tag{1.17}$$

to conclude that ∂ is the boundary operator on smooth chains C^∞_k. The homology of the smooth chain complex $C_\bullet(M) = (C^\infty_k, \partial)$ are identical to the de Rham cohomology, since both complexes are over $\mathbb{R}$

$$H_k(M) = H_k(C_\bullet(M)) = H^k_{dR}(M). \tag{1.18}$$

Earlier we proved that in case of simplicial chains the $H^\Delta_0(M)$ has dimension equal to the number of connected components $b_0(M)$ so using the algebraic topology relation between simplicial and smooth homology

$$b_0(M) = \dim H^\Delta_0(M) = \dim H_0(M) = H^0_{DR}(M). \tag{1.19}$$

In terms of the de Rham complex the interpretation of this result is the following: The $H^0_{DR}(M)$ describes the constant functions on M. If M has m-components then the constant functions on M can be constructed by specifying m constants c_i, one for each connected component of M

$$(c_1, ..., c_m) \in H^0(M), \quad c_i \in \mathbb{R} \tag{1.20}$$

The constants c_i are not related in any way, since smoothness is the local feature, defined on each component individually. Hence the dimension of $H_0(M)$ matches with the number of conneceted components for M.

1.3 Poincare duality

Theorem (Poincare duality): For compact smooth manifold M without boundary

$$H^p(M)^* = H^{n-p}(M). \tag{1.21}$$

Proof: We can define the non-degenerate paring on smooth compact M of dimension n

$$\Omega^p(M) \times \Omega^{n-p}(M) \to \mathbb{R} \;:\; (\omega, \mu) \mapsto \int_M \omega \wedge \mu. \tag{1.22}$$

The paring descends to the paring on cohomology. If both ω and μ are from cohomology i.e. they are closed forms up to shift by exact form then

$$(\omega + d\epsilon, \mu) - (\omega, \mu) = \int_M (\omega + d\epsilon) \wedge \mu - \int_M \omega \wedge \mu = \int_M d(\epsilon \wedge \omega) = \int_{\partial M} \epsilon \wedge \omega = 0. \tag{1.23}$$

Corrolary: For connected compact n-dimensional manifold M without boundary

$$H^0(M) = \mathbb{R} = H^n(M). \tag{1.24}$$

Example: We can use the Poincare duality to predict the cohomology of the Rieman surface Σ i.e. the compact, connected 2d smooth manifold. Being connected means that

$$H^0(\Sigma) = \mathbb{R}. \tag{1.25}$$

Being 2d compact means that

$$H^2(\Sigma) = H^0(\Sigma) = \mathbb{R} \tag{1.26}$$

while the non-degenerate antisymmetric paring on $H^1(\Sigma)$ requires it to be even-dimensional

$$H^1(\Sigma) = \mathbb{R}^{2g}, \;\; g \in \mathbb{Z}^{\geq 0} \tag{1.27}$$

The integer g is the *genus* of the Riemann surface.

1.4 Poincare lemma via homotopy*

Lemma: (Poincare) The de Rham cohomology of $\mathbb{R}^n$ are given by

$$H^0(\mathbb{R}^n) = \mathbb{R}, \quad H^k(\mathbb{R}^n) = 0, \;\; k > 0 \tag{1.28}$$

Proof: We can represent $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$. As we saw before the $\mathbb{R}$ is contractible i.e there exists a pair of maps

$$\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n : (x^i, t) \mapsto x^i, \quad i : \mathbb{R}^n \mapsto \mathbb{R}^n \times \mathbb{R} \;:\; x^i \mapsto (x^i, 0) \tag{1.29}$$

such that

$$\pi \circ i = \mathrm{id}_{\mathbb{R}^n}, \;\; i \circ \pi \simeq \mathrm{id}_{\mathbb{R}^{n+1}}. \tag{1.30}$$

The pullback maps i^*, π^* on differential forms similarly obey

$$i^* \circ \pi^* = \mathrm{id}_{\Omega^*(\mathbb{R}^n)}, \;\; \pi^* \circ i^* = \mathrm{id}_{\Omega^*(\mathbb{R}^{n+1})} + dK + Kd \simeq \mathrm{id}_{\Omega^*(\mathbb{R}^{n+1})} \tag{1.31}$$

The Homotopy K is defined in the following way: Every p-form on $\mathbb{R}^n_x \times \mathbb{R}_t$ can be uniquely decomposed as a linear combination of

$$\omega^p = \alpha^p_I(x,t)dx^I + dt \wedge \beta^{p-1}_J(x,t)dx^J \tag{1.32}$$

while the homotopy K is defined via

$$K\omega^p(x,t) = -\left[\int_0^t ds\; \beta^{p-1}_J(x,s)\right] dx^J. \tag{1.33}$$

We can check that

$$\begin{aligned}(\pi^* \circ i^* - \mathrm{id}_{\Omega^*(\mathbb{R}^{n+1})})\omega^p &= \pi^* \circ i^* \omega^p - \omega^p \\ &= \pi^* \alpha^p_I(x,0)dx^I - (\alpha^p_I(x,t)dx^I + dt \wedge \beta^{p-1}_J(x,t)dx^J) \\ &= \alpha^p_I(x,0)dx^I - \alpha^p_I(x,t)dx^I - dt \wedge \beta^{p-1}_J(x,t)dx^J,\end{aligned} \tag{1.34}$$

while the homotopy part

$$
\begin{aligned}
(dK+Kd)\omega^p &= dK\omega^p + Kd\omega^p \\
&= d\left[-\int_0^t ds\ \beta_J^{p-1}(x,s)\right] dx^J + Kd(\alpha_I^p(x,t)dx^I + dt\wedge\beta_J^{p-1}(x,t)dx^J) \\
&= -\int_0^t ds\ \partial_j\beta_J^{p-1}(x,s)dx^j\wedge dx^J - \beta_J(x,t)dt\wedge dx^J \\
&- \int_0^t ds\ \partial_s\alpha_I^p(x,s)dx^I + \int_0^t ds\partial_j\beta_J^{p-1}(x,s)dx^j\wedge dx^J \\
&= -\beta_J(x,t)dt\wedge dx^J - \alpha_I^p(x,t)dx^I + \alpha_I^p(x,0)dx^I \\
&= (\pi^*\circ i^* - \mathrm{id}_{\Omega^*(\mathbb{R}^{n+1})})\omega^p
\end{aligned}
\tag{1.35}
$$

an existence of homotopy completes proof of the Poincare lemma.

Remark: In our recursive proof we used $H^*(\mathbb{R}^n)$ to describe the $H^*(\mathbb{R}^n\times\mathbb{R})$. There exists a generalization of our construction for a Cartesian product of manifolds. For the pair of smooth manifolds X, Y the cohomology of the Cartesian product $X\times Y$ are given by the *Kunneth formula*

$$H^k(X\times Y) = \bigoplus_{p+q=k} H^p(X)\otimes H^q(Y). \tag{1.36}$$

The recursive Poincare lemma immediately follows from

$$H^0(\mathbb{R}) = \mathbb{R},\ \ H^1(\mathbb{R}) = 0 \quad \Longrightarrow H^p(X\times\mathbb{R}) = H^p(X)\otimes_{\mathbb{R}}\mathbb{R} = H^p(X). \tag{1.37}$$

1.5 Mayer-Vietoris construction*

Suppose $M = U\cup V$ with U and V being open. Then there is a sequence of inclusions

$$M \xleftarrow{\ \pi\ } U\sqcup V \underset{j}{\overset{i}{\leftleftarrows}} U\cap V \tag{1.38}$$

where we used i, j labels for natural inclusions $i: U\cap V\hookrightarrow U$ and $j: U\cap V\hookrightarrow V$. We can turn inclusion maps into maps for differential forms

$$\Omega^*(M) \xrightarrow{\ \pi^*\ } \Omega^*(U)\oplus\Omega^*(V) \underset{i^*}{\overset{j^*}{\rightrightarrows}} \Omega^*(U\cap V) \tag{1.39}$$

Using $f = i^* - j^*$ we can write an exact sequence

$$0 \longrightarrow \Omega^*(M) \xrightarrow{\pi^*} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{f} \Omega^*(U \cap V) \longrightarrow 0 \tag{1.40}$$

$$(u, v) \longrightarrow u - v$$

Proposition: Mayer-Vietoris sequence is exact (For the proof see Bott-Tu book).

Let us apply the cohomology functor to this short exact sequence and get the long exact sequence of cohomologies

$$\begin{array}{l} \ldots \longrightarrow H^k(M) \longrightarrow H^k(U) \oplus H^k(V) \xrightarrow{f} H^k(U \cap V) \\ \longrightarrow H^{k+1}(M) \longrightarrow H^{k+1}(U) \oplus H^{k+1}(V) \xrightarrow{f} H^{k+1}(U \cap V) \longrightarrow \ldots \end{array} \tag{1.41}$$

The same diagram, centered around $H^k(M)$

$$\ldots \longrightarrow H^{k-1}(U \cap V) \longrightarrow H^k(M) \xrightarrow{\pi^*} H^k(U) \oplus H^k(V) \xrightarrow{f} H^k(U \cap V) \longrightarrow \ldots \tag{1.42}$$

Example: We can construct an S^1 by from two open intervals I, J whose intersection is also two intervals U, V

$$S^1 = I \cup J, \quad I \cap J = U \sqcup V$$

The corresponding short exact sequence of forms

$$0 \longrightarrow \Omega^*(S^1) \xrightarrow{\pi^*} \Omega^*(I) \oplus \Omega^*(J) \xrightarrow{f} \Omega^*(U) \oplus \Omega^*(V) \longrightarrow 0$$

and long exact sequence of cohomologies

$$\begin{array}{l} 0 \longrightarrow H^0(S^1) \longrightarrow H^0(I) \oplus H^0(J) \xrightarrow{f} H^0(U) \oplus H^0(V) \\ \longrightarrow H^1(S^1) \longrightarrow H^1(I) \oplus H^1(J) \xrightarrow{f} H^1(U) \oplus H^1(V) \longrightarrow 0 \end{array}$$

The cohomologies of an open interval are the same as the ones for $\mathbb{R}$ i.e.

$$H^k(U) = H^k(V) = H^k(I) = H^k(J) = H^k(\mathbb{R}) = \begin{cases} \mathbb{R}, & k = 0; \\ 0, & k = 1. \end{cases} \tag{1.43}$$

Let us substitute the known terms in long exact sequence

$$0 \longrightarrow H^0(S^1) \longrightarrow \mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2 \longrightarrow H^1(S^1) \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$$

and simplify by dropping the trivial entries

$$0 \longrightarrow H^0(S^1) \longrightarrow \mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2 \longrightarrow H^1(S^1) \longrightarrow 0$$

We can immediately solve for $H^k(S^1)$ in terms of f.

$$H^0(S^1) = \ker f, \quad H^1(S^1) = \text{coker } f = \mathbb{R}^2/\text{Im } f \tag{1.44}$$

By construction

$$f : \mathbb{R}^2 \to \mathbb{R}^2 : \ (u, v) \mapsto \ u - v \tag{1.45}$$

so

$$H^0(S^1) = \ker f = \mathbb{R}, \quad H^1(S^1) = \text{coker } f = \mathbb{R}^2/\text{Im } f = \mathbb{R}^2/\mathbb{R} = \mathbb{R}. \tag{1.46}$$

The Mayer-Vietoris construction allows us to make several statements about cohomology

1. If $H^*(U)$ and $H^*(V)$ are finite-dimensional then $H^*(U \cup V)$ is also finite-dimensional.
2. We can iterate the MV construction till U_i become open discs in $\mathbb{R}^n$, which ave trivial cohomology. Thus we conclude that the cohomology $H^*(M)$ can be uniquely defined in terms of the structure of the finite open cover of M.

1.6 Cohomology as topological invariant

The de Rham cohomology are invariant under the infinitesimal diffeomorphisms. An infinitesimal diffeomorphism acts on differential forms as a Lie derivative

$$\delta_v \omega = \mathcal{L}_v \omega = d\iota_v \omega + \iota_v d\omega \tag{1.47}$$

For $\omega \in H^*(M)$ we have $d\omega = 0$ and is defined as a equivalnce class

$$[\omega] = [\omega + d\alpha] \tag{1.48}$$

so we conclude that

$$\delta_v[\omega] = [\delta_v \omega] = [d\iota_v \omega] = 0. \tag{1.49}$$

1.7 Hodge star

Definition: Riemann metric g is a bilinear positive-definite form on tangent space. In particular

$$g_p : T_pM \times T_pM \to \mathbb{R} \tag{1.50}$$

Let us choose the local coordinates $x^1, .., x^n$ around point p which induce the natural basis $\partial_1, .., \partial_n$ on the tangent T_pM space at p while the metric g can be described in terms of $n \times n$ matrix

$$g_{ij}(p) = g_p(\partial_i, \partial_j) \tag{1.51}$$

We will use the physics notation for the determinant of g

$$g = \det g_{ij} \in C^\infty(M). \tag{1.52}$$

Observation: We can extend the Riemann metric g to the positive definite paring on 1-forms T_p^*M using the inverse matrix g^{ij}

$$g^{ij} = g_p(dx^i, dx^j). \tag{1.53}$$

Furthermore we can extend the metric g on M to the paring on k-forms in natural way

$$g_p(\omega, \mu) = \omega_{i_1 ... i_k} \mu_{j_1 .. j_k} g^{i_1 j_1} ... g^{i_k j_k} \tag{1.54}$$

Definition: Given a metric g on n-dimensional manifold M we can define a linear map

$$* : \ \Omega^p(M) \to \Omega^{n-p}(M) \ : \ \omega \mapsto *\omega \tag{1.55}$$

as unique form $*\omega$ such that

$$\mu \wedge *\omega = g(\mu, \omega) \cdot \mathrm{vol}_g, \ \ \forall \mu \in \Omega^{n-p}(M) \tag{1.56}$$

where vol_g is the volume form constructed from metric g. The $*\omega$ is the *Hodge dual form* for differential form ω. In local coordinates $x^1, .., x^n$ the volume form

$$\mathrm{vol}_g = \sqrt{g}\, dx^1 \wedge ... \wedge dx^n. \tag{1.57}$$

In components

$$\omega = \frac{1}{k!}\sum \omega_{i_1..i_k} dx^{i_1}....dx^{i_k}, \tag{1.58}$$

while the dual form

$$* \omega = \frac{1}{(n-k)!}\sum (*\omega)_{j_1...j_{n-k}} dx^{j_1}....dx^{j_{n-k}}, \tag{1.59}$$

with

$$(*\omega)_{j_{k+1}...j_n} = \frac{\sqrt{g}}{k!}\epsilon_{j_1...j_n} g^{i_1 j_1}...g^{i_k j_k}\omega_{i_1..i_k}. \tag{1.60}$$

The square of the Hodge dual is the identity up to a sign. In particular for $n = \dim M$ we have

$$* *\omega = (-1)^{k(n-k)}\omega, \ \ \omega \in \Omega^k(M). \tag{1.61}$$

Example: Let x^1, x^2 be coordinates on $\mathbb{R}^2$ with Euclidean metric $g_{ij} = \delta_{ij}$ then

$$* 1 = \frac{1}{2!}\sum (*1)_{ij} dx^i \wedge dx^j = \frac{1}{2}\sum \epsilon_{ij} dx^i \wedge dx^j = dx^1 \wedge dx^2. \tag{1.62}$$

The 1-forms

$$* dx^1 = \frac{1}{1!}\sum (*dx^1)_i dx^i = \sum \frac{1}{1!}\epsilon_{ji} g^{j1} dx^i = \epsilon_{1i} dx^i = \epsilon_{12} dx^2 = dx^2 \tag{1.63}$$

$$* dx^2 = \frac{1}{1!}\sum (*dx^2)_i dx^i = \sum \frac{1}{1!}\epsilon_{ji} g^{j2} dx^i = \epsilon_{2i} dx^i = \epsilon_{21} dx^1 = -dx^1 \tag{1.64}$$

and the top form

$$* dx^1 \wedge dx^2 = \frac{1}{0!}(*dx^1 \wedge dx^2) = \sum \frac{1}{2!}\epsilon_{ij} g^{i1} g^{j2} = \epsilon_{12} = 1. \tag{1.65}$$

1.8 Paring on differential forms

For a compact M we with Riemann metric we can define a positive definite paring

$$\Omega^p(M) \times \Omega^p(M) \to \mathbb{R}^{\geq 0} \ : \ (\omega, \mu) \mapsto \langle \omega, \mu \rangle = \int_M \omega \wedge *\mu \geq 0, \tag{1.66}$$

what turn the pair $(\Omega^*(M), \langle \cdot, \cdot \rangle)$ into a *Hilbert space*. We can define the conjugated operator

$$\langle d\omega, \mu \rangle = \langle \omega, d^*\mu \rangle, \ \ \forall \omega \in \Omega^{p-1}(M). \tag{1.67}$$

Using the the seven-names theorem we can express d^* in terms of external derivative and Hodge star

$$d^* = (-1)^{np+n+1} * d* : \quad \Omega^p(M) \to \Omega^{p-1}(M). \tag{1.68}$$

Indeed

$$\begin{aligned} 0 &= \int_M d(\omega \wedge *\mu) = \int_M (d\omega \wedge *\mu + (-1)^{p-1}\omega \wedge d * \mu) \\ &= \langle d\omega, \mu\rangle + (-1)^{p-1} \int_M \omega \wedge (-1)^{(n-p+1)(n-(n-p+1))} * *d * \mu \\ &= \langle d\omega, \mu\rangle + (-1)^{p-1}(-1)^{(n-p+1)(p-1)}\langle\omega, *d * \mu\rangle \\ &= \langle d\omega, \mu\rangle + (-1)^{(n-p)(p-1)}\langle\omega, *d * \mu\rangle = \langle d\omega, \mu\rangle + (-1)^{n(p-1)}\langle\omega, *d * \mu\rangle \\ &= \langle d\omega, \mu\rangle - (-1)^{np+n+1}\langle\omega, *d * \mu\rangle, \end{aligned} \tag{1.69}$$

where we used

$$p(p-1) \in 2\mathbb{Z}, \quad \forall p \in \mathbb{Z}. \tag{1.70}$$

Using d and d^* we can construct a second order differential operator

$$\Delta = dd^* + d^*d = \{d, d^*\} : \Omega^p(M) \to \Omega^p(M) \tag{1.71}$$

also known as the *Hodge-Laplacian.*

Example: Hodge-Laplacian on functions matches with the usual Laplacian

$$\begin{aligned} \Delta f &= (dd^* + d^*d)f = d^*df = (-1)^{np+n+1} * d * df = (-1)^{2n+1} * d * df \\ &= - * d * df = - * d * \sum \partial_i f dx^i \\ &= - * d \frac{1}{(n-1)!} \sum \sqrt{g}\epsilon_{jj_1...j_{n-1}} g^{ij}\partial_i f dx^{j_1} \wedge ... \wedge dx^{j_{n-1}} \\ &= - * \frac{1}{(n-1)!} \sum \partial_k(\sqrt{g}g^{ij}\partial_i f)\epsilon_{jj_1...j_{n-1}} dx^k \wedge dx^{j_1} \wedge ... \wedge dx^{j_{n-1}} \\ &= -\sum \frac{n\sqrt{g}}{n!}\partial_k(\sqrt{g}g^{ij}\partial_i f)\epsilon_{jj_1...j_{n-1}}\epsilon_{lk_1....k_{n-1}} g^{kl} g^{k_1j_1} ... g^{k_{n-1}j_{n-1}} \\ &= -\frac{\sqrt{g}}{g}\sum \delta^k_j \partial_k(\sqrt{g}g^{ij}\partial_i f) = -\frac{1}{\sqrt{g}}\sum \partial_j(\sqrt{g}g^{ij}\partial_i f), \end{aligned} \tag{1.72}$$

where $n = \dim M$. Hodge-Laplacian on functions is identical to the covariant Laplacian

$$\Delta f = g^{ij}\nabla_i\nabla_j f. \tag{1.73}$$

Hodge-Laplacian operator has several nice properties:

- Δ commutes with d and d^*

$$d\Delta = \Delta d, \quad d^*\Delta = \Delta d^*. \tag{1.74}$$

- Δ is self-dual i.e.

$$\Delta^* = \Delta. \tag{1.75}$$

- Δ has bounded spectrum on compact manifold

$$\langle \omega, \Delta\omega \rangle = \langle \omega, (dd^* + d^*d)\omega \rangle = \langle d\omega, d\omega \rangle + \langle d^*\omega, d^*\omega \rangle \geq 0 \tag{1.76}$$

- d and d^* act trivially on $\ker \Delta$, what immediately follows from

$$0 = \langle \omega, \Delta\omega \rangle = \langle d\omega, d\omega \rangle + \langle d^*\omega, d^*\omega \rangle \Longrightarrow d\omega = d^*\omega = 0 \tag{1.77}$$

1.9 Hodge Theory

Observation: An existence of Hodge-Lapalacian operator allows us to to "invert" the differential d. Let us consider a closed differential form ω, which does not belong to $\ker \Delta$

$$d\omega = 0, \ \ \omega \in \Omega^*(M) \setminus \ker \Delta. \tag{1.78}$$

For such ω we can write

$$\Delta\omega = (dd^* + d^*d)\omega = dd^*\omega = d(d^*\omega). \tag{1.79}$$

Since ω is outside of the $\ker \Delta$ we can invert the Laplacian

$$\omega = \Delta^{-1} d(d^*\omega) = d\left(\frac{d^*}{\Delta}\omega\right). \tag{1.80}$$

As we saw before the Hodge-Laplacian is non-negative define so we can use the integral representation to write

$$\omega = d\int_0^\infty dt\ e^{-t\Delta} d^*\omega. \tag{1.81}$$

We observed that closed forms outside the $\ker \Delta$ are all exact forms, so de Rham cohomology should be among the harmonic forms. We can formalize and strengthen our observation in the form of Hodge theorem and prove it using the homotopy K, similar to our expression for inverse of d.

Theorem (Hodge): On compact closed manifold M with Riemann metric g de Rham cohomology are isomorphic to the *harmonic forms*

$$H^*_{dR}(M) = \ker\Delta, \quad \Delta = dd^* + d^*d. \tag{1.82}$$

Proof: On a compact manifold, where Δ has bounded spectrum we can construct homotopy

$$K = \int_0^\infty dt\, e^{-t\Delta}d^* :\ \Omega^p(M) \to \Omega^{p-1}(M) \tag{1.83}$$

which obeys

$$\{d, K\} = dK + Kd = \int_0^\infty dt\Delta e^{-t\Delta} = \int_0^\infty -\partial_t(e^{t\Delta}) = 1 - \lim_{t\to\infty} e^{-t\Delta} \tag{1.84}$$

The limiting term is the projection on $\ker\Delta \subset \Omega^*(M)$

$$\Pi_0 : \Omega^*(M) \to \ker\Delta, \quad \Pi_0 = \lim_{t\to\infty} e^{-t\Delta}, \quad \Pi_0^2 = \Pi_0 \tag{1.85}$$

Let us consider a pair of complexes, the de Rham complex

$$D^\bullet = \qquad \Omega^0(M)\xrightarrow{d}\ldots.\xrightarrow{d}\Omega^k(\mathbb{R})\xrightarrow{d}\Omega^{k+1}(\mathbb{R})\xrightarrow{d}\ldots\longrightarrow\Omega^n(M)\xrightarrow{d}0 \tag{1.86}$$

and complex of harmonic forms

$$C^\bullet = \qquad \Omega^0_\Delta(M)\xrightarrow{0}\ldots.\xrightarrow{0}\Omega^k_\Delta(\mathbb{R})\xrightarrow{0}\Omega^{k+1}_\Delta(\mathbb{R})\xrightarrow{0}\ldots\longrightarrow\Omega^n_\Delta(M)\xrightarrow{0}0 \tag{1.87}$$

with $\Omega^k_\Delta(M)$ being harmonic k-forms and zero differential. There are pair of chain maps

$$\pi : D^\bullet \to C^\bullet, \quad i : D^\bullet \hookrightarrow C^\bullet, \tag{1.88}$$

which are inverse of each other up to a homotopy

$$\pi \circ i = \mathrm{id}_{C^\bullet}, \quad i \circ \pi = \Pi_0 = \mathrm{id}_{C^\bullet} - dK - Kd. \tag{1.89}$$

Hence the two complexes $C^\bullet$ and $D^\bullet$ have identical cohomology

$$H^*_{dR} = H^*(D^\bullet) = H^*(C^\bullet) = \Omega^*_\Delta(M) = \ker\Delta, \tag{1.90}$$

while the cohomology for harmonic forms complex $C^\bullet$ are identical to the vector spaces in complex

$$H^k(C^\bullet) = \Omega^k_\Delta(M) = \ker\Delta \cap \Omega^k(M). \tag{1.91}$$

Example: You might familiar from calculus that the only solutions to Laplacian equation for functions on a compact manifold are constant functions. If manifold has n connected components the space of solutions is

$$\ker\Delta \cap \Omega^0(M) = \mathbb{R}^n = H^0(M). \tag{1.92}$$

1.10 Euler characteristic for de Rham complex*

In previous sections we defined an Euler characteristic for the chain complex

$$\chi(C_\bullet) = \sum (-1)^k \dim C_k. \tag{1.93}$$

Let us consider an Euler characteristic of the de Rham complex

$$\chi(M) = \sum_{k=1}^{n} (-1)^k \dim\Omega^k(M). \tag{1.94}$$

Let us further assume that M is closed manifold endowed with Riemann metric on M, so we can assign a symmetric positive-definite paring on differential forms

$$\langle\omega,\mu\rangle = \int \omega\wedge *\mu. \tag{1.95}$$

Using this paring we can chose an orthonormal basis $\{\omega_\alpha\}$ in Ω^* and define *trace* on operators

$$\mathcal{O} : \Omega^*(M) \to \Omega^*(M) \tag{1.96}$$

$$\mathrm{Tr}\mathcal{O} \equiv \sum_{\omega_\alpha} \langle\omega_\alpha, \mathcal{O}\omega_\alpha\rangle = \sum_{\omega_\alpha} \int \mathcal{O}\omega_\alpha \wedge *\mathcal{O}\omega_\alpha. \tag{1.97}$$

The trace is invariant under cyclic permutation

$$\begin{aligned}\mathrm{Tr}(\mathcal{O}_A\mathcal{O}_B\mathcal{O}_C) &= \sum_{\omega_\alpha}\langle\omega_\alpha, \mathcal{O}_A\mathcal{O}_B\mathcal{O}_C\omega_\alpha\rangle \\ &= \sum_{\omega_\alpha,\omega_\beta,\omega_\gamma}\langle\omega_\alpha, \mathcal{O}_A\omega_\beta\rangle\langle\omega_\beta, \mathcal{O}_B\omega_\gamma\rangle\langle\omega_\gamma, \mathcal{O}_C\omega_\alpha\rangle \\ &= \sum_{\omega_\alpha,\omega_\beta,\omega_\gamma}\langle\omega_\gamma, \mathcal{O}_C\omega_\alpha\rangle\langle\omega_\alpha, \mathcal{O}_A\omega_\beta\rangle\langle\omega_\beta, \mathcal{O}_B\omega_\gamma\rangle \\ &= \sum_{\omega_\alpha}\langle\omega_\alpha, \mathcal{O}_C\mathcal{O}_A\mathcal{O}_B\omega_\alpha\rangle = \mathrm{Tr}(\mathcal{O}_C\mathcal{O}_A\mathcal{O}_B).\end{aligned} \tag{1.98}$$

We can express Euler characteristic as a trace

$$\chi(M) = \mathrm{Tr}(-1)^F, \tag{1.99}$$

where the operator $(-1)^F$ originated from physics literature and defined via

$$(-1)^F:\ \Omega^p(M) \to \Omega^p(M):\ \omega \mapsto (-1)^p\omega. \tag{1.100}$$

The trace formula require a regularization, what can be seen from the simple case of $M = \mathbb{R}^n$. We can represent the differential forms as tensor product

$$\Omega^*(\mathbb{R}^n) = \Lambda^*\mathbb{R}^n \otimes C^\infty(\mathbb{R}^n), \tag{1.101}$$

while the $(-1)^F$ operator is diagonal in this decomposition

$$(-1)^F = (-1)^F \otimes \mathrm{id} : \Lambda^*\mathbb{R}^n \otimes C^\infty(\mathbb{R}^n) \to \Lambda^*\mathbb{R}^n \otimes C^\infty(\mathbb{R}^n). \tag{1.102}$$

The trace over the space of forms $\Lambda^*\mathbb{R}^n$ is

$$\mathrm{Tr}_{\Lambda^*\mathbb{R}^n}(-1)^F = \sum_{k=0}^{n}(-1)^k \dim(\Lambda^k\mathbb{R}^n) = \sum_{k=0}^{n}(-1)^k C_n^k = (1-1)^n = 0, \tag{1.103}$$

while the Euler characteristics

$$\chi(M) = \mathrm{Tr}_{\Omega^*(\mathbb{R}^n)}(-1)^F = \mathrm{Tr}_{\Lambda^*\mathbb{R}^n}(-1)^F \cdot \dim C^\infty(\mathbb{R}^n) = 0\cdot\infty \tag{1.104}$$

is not well defined and require some regularization procedure.

Following physics literature we can introduce *proper-time regularization*, which relies on non-negative spectrum of Laplacian

$$\chi_\beta(M) = \mathrm{Tr}(-1)^F e^{-\beta\Delta}, \quad \chi(M) = \lim_{\beta\to 0} \chi_\beta(M). \tag{1.105}$$

We can express the regulator in the form

$$e^{-\beta\Delta} = 1 - \beta\Delta + ... = 1 - \beta(dd^* + d^*d) + ... = 1 + dK_\beta + K_\beta d \tag{1.106}$$

for homotopy

$$K_\beta = -\int_0^\beta e^{-t\Delta} d^* \quad : \quad \Omega^p(M) \mapsto \Omega^{p-1}(M). \tag{1.107}$$

Let us observe that the both homotopy K_β and differential d change degree by one so

$$(-1)^F d = -d(-1)^F, \quad (-1)^F K_\beta = -K_\beta(-1)^F. \tag{1.108}$$

Using cyclicity of trace we can show that

$$\begin{aligned} \mathrm{Tr}[(-1)^F(K_\beta d + dK_\beta)] &= -\mathrm{Tr}[K_\beta(-1)^F d] - \mathrm{Tr}[d(-1)^F K_\beta] \\ &= -\mathrm{Tr}[dK_\beta(-1)^F] - \mathrm{Tr}[K_\beta d(-1)^F] \\ &= -\mathrm{Tr}[(-1)^F dK_\beta] - \mathrm{Tr}[(-1)^K K_\beta d] \\ &= -\mathrm{Tr}[(-1)^F(K_\beta d + dK_\beta)] = 0. \end{aligned} \tag{1.109}$$

The regularized Euler characteristic is independent of β so

$$\chi_\beta(M) = \mathrm{Tr}(-1)^F e^{-\beta\Delta} = \mathrm{Tr}[(-1)^F(1 + K_\beta d + dK_\beta)] = \mathrm{Tr}(-1)^F = \chi(M). \tag{1.110}$$

Since the expression $\chi_\beta(M)$ is independent of β we can take $\beta \to \infty$ limit, where the regularization become projector on harmonic forms

$$\lim_{\beta\to 0} e^{-\beta\Delta} = \Pi_0. \tag{1.111}$$

By Hodge theorem Harmonic forms are the same as de Rham cohomology, hence vector spaces $\Omega^k_\Delta(M)$ are finite-dimensional so we can express

$$\chi(M) = \lim_{\beta\to\infty} \chi_\beta(M) = \sum_{k=0}^n (-1)^k \dim \Omega_\Delta(M). \tag{1.112}$$

Equivalently we can replace $\Omega^k_\Delta(M)$ by de Rham cohomology $H^k_{dR}(M)$

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \dim \Omega_\Delta(M) = \sum_{k=1}^{n} (-1)^k \dim H^k_{dR}(M) = \sum_{k=0}^{n} (-1)^k b_k(M), \tag{1.113}$$

to derive the infinite-dimensional version of the Euler characteristic formula for complexes.