Supersymmetric Quantum Mechanics and Morse theory

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Lectures 3,4

Chain complexes. Homology. Homotopy.

Contents

Contents

1 Cohomology theory

1.1 Chain complex

Definition: A *chain complex* $C_\bullet$, which is a sequence of abelian groups C_k and homomorphisms $\partial_k : C_k \to C_{k-1}$

$$C_\bullet = \ldots \longrightarrow C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} C_{k-2} \longrightarrow \ldots . \tag{1.1}$$

Homomorphisms ∂_k are called boundary maps if

$$\partial_k \circ \partial_{k+1} = 0, \quad \forall k. \tag{1.2}$$

The above relation implies that

$$\operatorname{Im} \partial_k \subseteq \ker \partial_{k-1}. \tag{1.3}$$

Remark: The equality is the special case of complex known as *exact sequence*.

Definition: Complex $(C_\bullet, \partial)$ is the *exact sequence* when

$$\operatorname{Im} \partial_k = \ker \partial_{k-1}, \quad \forall k. \tag{1.4}$$

1.2 Homology

The *homology groups of the chain complex* $C_\bullet$ are defined as

$$H_k(C) = \frac{Z_k(C)}{B_k(C)}, \tag{1.5}$$

where

$$Z_k(C) = \ker(\partial_k : C_k \to C_{k-1}) \tag{1.6}$$

is a set of closed k-chains, denoted as k-*cycles* and

$$B_k(C) = \mathrm{Im}(\partial_{k+1} : C_{k+1} \to C_k) \tag{1.7}$$

is a set of exact k-chains, denoted as k-*boundaries*.

Remark: Homology of exact sequence are trivial. Essentially the homology of a chain complex measure how much it fails to be an exact sequence.

1.3 Cohomology

We can define *cochain groups* C^k as dual to the chain groups

$$C^k = Hom(C_k, \mathbb{R}) = C_k^*. \tag{1.8}$$

Moreover, cochain groups form cochain complex $C^\bullet = (C^k, d^k)$ with coboundary maps

$$d^k : C^k \to C^{k+1}, \tag{1.9}$$

defined via

$$(d^k f)(c) = f(\partial_{k+1} c), \quad \forall c \in C_{k+1}. \tag{1.10}$$

With this definition it is not hard to show that

$$d^{k+1} \circ d^k = 0. \tag{1.11}$$

The coboundary map d is often called the *differential*.

The construction of cochain complex can be summarized by the diagram

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & C_{k+1} & \xrightarrow{\partial_{k+1}} & C_k & \xrightarrow{\partial_k} & C_{k-1} & \xrightarrow{\partial_{k-1}} & C_{k-2} & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow {\scriptstyle Hom(*,\mathbb{R})} & & \downarrow & \\
\cdots \longleftarrow & C^{k+1} & \xleftarrow[d^k]{} & C^k & \xleftarrow[d_{k-1}]{} & C^{k-1} & \xleftarrow[d_{k-2}]{} & C^{k-2} & \longleftarrow \qquad \cdots
\end{array}
$$

The *cohomology groups* are defined as

$$H^k(C) = \frac{Z^k(C)}{B^k(C)}, \tag{1.12}$$

where

$$Z^k(C) = \ker(d^k : C^k \to C^{k+1}) \tag{1.13}$$

is a set of closed k-cochains, denoted as k-*cocycles* and

$$B^k(C) = \mathrm{Im}(d^{k-1} : C^{k-1} \to C^k) \tag{1.14}$$

is a set of exact k-cochains, denoted as k-*coboundaries*.

Remark: In a literature the generic term *complex* describes cochain type of complex i.e with differential raising degree by one.

1.4 Euler characteristic

We can define the Euler characteristic for arbitrary simplicial model. Let us recall the 1d and 2d definitions of Euler characteristics

$$\chi(\Delta_{M^1}) = V - E = \dim C_0 - \dim C_1 \tag{1.15}$$

$$\chi(\Delta_{M^2}) = V - E + F = \dim C_0 - \dim C_1 + \dim C_2 \tag{1.16}$$

Natural generalization to arbitrary simplicial complex

$$\chi(C_\bullet) = \dim C_0 - \dim C_1 + \dim C_2 + ... + (-1)^n \dim C_n = \sum_{k=0}^{n} (-1)^k \dim C_k. \tag{1.17}$$

Proposition: We can express Euler characteristic for complex $C_\bullet$ in terms of cohomology $H_k(C_\bullet)$

$$\chi(C_\bullet) = \sum_{k=0}^{n} (-1)^k \dim C_k. \tag{1.18}$$

Proof: We can use the *isomorphism theorem* for linear map $f : V \to W$ between two vector spaces V, W to establish the isomorphism

$$\mathrm{Im} f = V / \ker f, \tag{1.19}$$

which implies that

$$\begin{aligned} \dim(\mathrm{Im} f) &= \dim(V/\ker f) = \dim V - \dim \ker f \\ \dim V &= \dim(\ker f) + \dim(\mathrm{Im} f). \end{aligned} \tag{1.20}$$

Chain complex

$$C_\bullet = \ldots \longrightarrow C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} C_{k-2} \longrightarrow \ldots. \tag{1.21}$$

is a sequence of maps $\partial_k : C_k \to C_{k-1}$, so we can express

$$\dim C_k = \dim(\ker \partial_k) + \dim(\mathrm{Im} \partial_k). \tag{1.22}$$

The Euler characteristic of complex $C_\bullet$

$$\begin{aligned} \chi(C_\bullet) &= \dim C_0 - \dim C_1 + \dim C_2 + \ldots + (-1)^n \dim C_n \\ &= \dim \ker \partial_0 + \dim \mathrm{Im} \partial_0 - \dim \ker \partial_1 - \dim \mathrm{Im} \partial_1 + \dim \ker \partial_2 + \dim \mathrm{Im} \partial_2 + \ldots \\ &= \dim \mathrm{Im} \partial_0 + (\dim \ker \partial_0 - \dim \mathrm{Im} \partial_1) - (\dim \ker \partial_1 - \dim \mathrm{Im} \partial_2) + \ldots \\ &= \dim H_0(C) - \dim H_1(C) + \dim H_2(C) + .. + (-1)^n \dim H_n(C) \\ &= b_0 - b_1 + b_2 + \ldots + (-1)^n b_n = \sum_{k=0}^{n} (-1)^k b_k \end{aligned} \tag{1.23}$$

In the equality we used

$$\dim \mathrm{Im} \partial_0 = 0, \tag{1.24}$$

since $\partial_0 = 0$ and

$$H_n(C) = \ker \partial_n. \tag{1.25}$$

1.5 Simplicial (co)homology

Given a manifold M we can construct its simplicial model Δ_M and simplicial complex $C_\bullet = (C_k(\Delta_M), \partial)$. The homologies $H_k(C)$ of simplicial complex are known as the *simplicial homology* of M often denoted as $H_n^\Delta(M)$.

Example: The simplest manifold is a point $M = pt$ with a triangulation being a single 0-simplex

$$C_0 = \mathbb{R}\langle e_0 \rangle = \mathbb{R}. \tag{1.26}$$

The simplicial complex is

$$C_\bullet(\Delta_0) = \mathbb{R}^1 \xrightarrow{\partial_0} 0. \tag{1.27}$$

The homology groups of such chain complex are

$$H_0^\Delta(pt) = H_0(C) = \ker \partial_0 = \mathbb{R}. \tag{1.28}$$

The corresponding cochain complex

$$C^\bullet(\Delta_0) = 0 \xleftarrow[d_0]{} \mathbb{R}^1. \tag{1.29}$$

The cohomology of $C^\bullet$

$$H^0_\Delta(I) = H^0(C) = \ker d_0 = \mathbb{R}. \tag{1.30}$$

Example: Let us look at the simplicial model for an interval $I = [0, 1]$ with chain complex

$$C_\bullet = \ \mathbb{R} \xrightarrow{\partial_1} \mathbb{R}^2 \xrightarrow{\partial_0} 0 \tag{1.31}$$

$$c_{01} \longrightarrow (-c_{01}, c_{01})$$

The homology groups of such chain complex are

$$H_0^\Delta(I) = H_0(C) = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \mathbb{R}^2/\mathbb{R}^1 = \mathbb{R} \tag{1.32}$$

and

$$H_1^\Delta(I) = H_1(C) = \ker \partial_1 = \{0\}. \tag{1.33}$$

The corresponding cochain complex

$$C^\bullet = 0 \xleftarrow[d_1]{} \mathbb{R} \xleftarrow[d_0]{} \mathbb{R}^2 \tag{1.34}$$

$$a^0 - a^1 \longleftarrow (a^0, a^1)$$

We used the definition

$$d_0 : C^0 \to C^1, \quad (d_0 f)(c) = f(\partial_1 c) \tag{1.35}$$

to evaluate

$$(d_0 f^0)(c) = (d_0 f^0)(c_{01} e_{01}) = f^0(c_{01} e_0 - c_{01} e_1) = c_{01} \implies (d_0 f^0)(e_{01}) = f^{01}(e_{01}) \tag{1.36}$$

and

$$(d_0 f^1)(c) = (d_0 f^1)(c_{01} e_{01}) = f^1(c_{01} e_0 - c_{01} e_1) = -c_{01} \implies (d_0 f^1)(e_{01}) = -f^{01}(e_{01}). \tag{1.37}$$

The cohomology of $C^\bullet$

$$H^0_\Delta(I) = H^0(C) = \ker d_0 = \mathbb{R} \tag{1.38}$$

and

$$H^1_\Delta(I) = H^1(C) = \frac{\ker d_1}{\text{Im } d_0} = \mathbb{R}/\mathbb{R} = \{0\}. \tag{1.39}$$

Example: Let us look at the simplicial model for a circle S^1, constructed from three 1d simplexes. The vector spaces

$$C_0 = \mathbb{R}\langle e_0, e_1, e_2 \rangle = \mathbb{R}^3, \;\; C_1 = \mathbb{R}\langle e_{01}, e_{12}, e_{20} \rangle = \mathbb{R}^3, \tag{1.40}$$

while the corresponding chain complex

$$C_\bullet = \quad \mathbb{R}^3 \xrightarrow{\partial_1} \mathbb{R}^3 \xrightarrow{\partial_0} 0 \tag{1.41}$$

$$(c_{01}, c_{12}, c_{20}) \longrightarrow (c_{20} - c_{01}, c_{01} - c_{12}, c_{20} - c_{12})$$

The homology groups of such chain complex are

$$H_0^\Delta(I) = H_0(C) = \frac{\ker \partial_0}{\text{Im } \partial_1} = \mathbb{R}^3/\mathbb{R}^2 = \mathbb{R} \tag{1.42}$$

and

$$H_1^{\Delta}(I) = H_1(C) = \ker \partial_1 = \mathbb{R}. \tag{1.43}$$

The corresponding cochain complex

$$C^{\bullet} = 0 \xleftarrow{d_1} \mathbb{R}^3 \xleftarrow{d_0} \mathbb{R}^3 \tag{1.44}$$

$$(a-b, b-c, c-a) \longleftarrow (a,b,c).$$

has cohomology

$$H^0_{\Delta}(I) = H^0(C) = \ker d_0 = \mathbb{R} \tag{1.45}$$

and

$$H^1_{\Delta}(I) = H^1(C) = \frac{\ker d_1}{\operatorname{Im} d_0} = \mathbb{R}^3/\mathbb{R}^2 = \mathbb{R}. \tag{1.46}$$

Remark*: In our examples

$$H^k_{\Delta}(C) = Hom(H_k^{\Delta}(C), \mathbb{R}) = H_k^{\Delta}(C)^*. \tag{1.47}$$

So the whole construction of cochain complex seems redundant as we can evaluate its cohomology by taking dual of the homology. However it is only true in case of (co)homology with coefficients in $\mathbb{R}$. In general can we can define complex with coefficients in any ring R, with the homology $H_*(C, R)$ with coefficients in ring R. Similarly we can define the cochain complex $C^{\bullet} = Hom(C_{\bullet}, R)$

$$H^k(C^{\bullet}) \neq Hom(H_k(C_{\bullet}), R). \tag{1.48}$$

while there exists a relation between two in the form of the *universal coefficients theorem*

$$0 \longrightarrow Ext(H_{k-1}(C_{\bullet}), R) \longrightarrow H^k(C^{\bullet}) \longrightarrow Hom(H_k(C_{\bullet}), R) \longrightarrow 0.$$

Example: Let us consider complex with two nontrivial entries

$$C_k = \begin{cases} \mathbb{Z}, & k = n-1; \\ \mathbb{Z}, & k = n; \\ 0, & k \neq n, n-1. \end{cases} \tag{1.49}$$

and boundary map $\partial_n : C_n \to C_{n-1}$ being multiplication by $m \in \mathbb{Z}$. We can summarize all information in the diagram

$$C_\bullet = \ldots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \longrightarrow 0 \longrightarrow \ldots. \tag{1.50}$$

The homology groups of such chain complex are trivial away from H_n and H_{n-1} while

$$H_n = \frac{\ker \partial_n}{\mathrm{Im}\ \partial_{n+1}} = 0, \quad H_{n-1} = \frac{\ker \partial_{n-1}}{\mathrm{Im}\ \partial_n} = \frac{\mathbb{Z}}{m\mathbb{Z}} \simeq \mathbb{Z}_m \tag{1.51}$$

$$H_k(C_\bullet) = \begin{cases} \mathbb{Z}_m, & k = n-1; \\ 0, & k \neq n. \end{cases} \tag{1.52}$$

The corresponding cochain groups

$$C^k = \begin{cases} \mathbb{Z}, & k = n-1; \\ \mathbb{Z}, & k = n; \\ 0, & k \neq n, n-1. \end{cases} \tag{1.53}$$

with the differential $d^{n-1} : C^{n-1} \to C^n$

$$(d^{n-1} f)(c) = f(\partial_n c) = f(mc) = mf(c) \tag{1.54}$$

so d^{n-1} is a multiplication by m as well. Let us summarize all info in terms of a diagram

$$C^\bullet = \ldots \longleftarrow 0 \longleftarrow \mathbb{Z} \xleftarrow[\times m]{} \mathbb{Z} \longleftarrow 0 \longleftarrow \tag{1.55}$$

has cohomologies

$$H^k(C^\bullet) = \begin{cases} \mathbb{Z}_m, & k = n; \\ 0, & k \neq n. \end{cases} \tag{1.56}$$

The dual of the homology

$$Hom(H_n(C_\bullet), \mathbb{Z}) = Hom(0, \mathbb{Z}) = 0 \neq \mathbb{Z}_m = H^n(C^\bullet) \tag{1.57}$$

while the universal coefficient theorem for our example predicts

$$0 \longrightarrow Ext(H_{n-1}(C_\bullet), \mathbb{Z}) \longrightarrow H^n(C^\bullet) \longrightarrow Hom(H_n(C_\bullet), \mathbb{Z}) \longrightarrow 0.$$

The Ext can be evaluated

$$Ext(H_{n-1}(C_\bullet), \mathbb{Z}) = Ext(\mathbb{Z}_m, \mathbb{Z}) = \mathbb{Z}_m \tag{1.58}$$

so the whole cohomology group in our example originate from Ext!

1.6 Direct sum

Definition: For a pair of complexes $C_\bullet$ and $D_\bullet$ we can define a *direct sum of complexes* $C_\bullet \oplus D_\bullet$ as a complex with

- Sequence of vector spaces
$$(C \oplus D)_n = C_n \oplus D_n. \tag{1.59}$$
- Linear maps
$$\partial_n : (C \oplus D)_n \to (C \oplus D)_{n-1} : (c, d) \mapsto (\partial^C c, \partial^D d). \tag{1.60}$$
- Differential property
$$\partial_{n-1} \circ \partial_n = (\partial^C \circ \partial^C, \partial^D \circ \partial^D) = 0. \tag{1.61}$$

Proposition: For a decomposition of M into disjoint union of M_α there is an isomorphism

$$H_n^\Delta(\sqcup M_\alpha) = \bigoplus_\alpha H_n^\Delta(M_\alpha). \tag{1.62}$$

Proof: The simplicial models follow the disjoint union i.e

$$\Delta_{M \sqcup N} = \Delta_M \sqcup \Delta_N. \tag{1.63}$$

The complex $C_\bullet(\Delta_{M \sqcup N})$ is the direct sum of complexes $C_\bullet(\Delta_M)$ and $C_\bullet(\Delta_N)$ so the cohomology

$$\begin{aligned} H_n^\Delta(M \sqcup N) &= H_n(C_\bullet(\Delta_{M \sqcup N}), \partial) = H_n(C_\bullet(\Delta_M) \oplus C_\bullet(\Delta_N), \partial) \\ &= H_n(C_\bullet(\Delta_M), \partial) \oplus H_n(C_\bullet(\Delta_N), \partial) = H_n^\Delta(M) \oplus H_n^\Delta(N). \end{aligned} \tag{1.64}$$

Proposition: If M is path-connected and non-empty then $H_0^\Delta(M) = \mathbb{R}$. Hence for any space $H_0^\Delta(M)$ has dimension equal to the number of connected components.

Proof: By construction

$$H_0^{\Delta}(M) = C_0(\Delta_M)/\text{Im}\partial_1. \tag{1.65}$$

Let us define a linear map

$$\varepsilon : C_0 \to \mathbb{R} \ : \ c_i e_i \mapsto \sum c_i. \tag{1.66}$$

The cohomology statement immediately follows from

$$\ker \varepsilon = \text{Im } \partial_1, \tag{1.67}$$

since

$$H_0^{\Delta}(M) = C_0(\Delta_M)/\text{Im}\partial_1 = H_0^{\Delta}(M) = C_0(\Delta_M)/\ker \varepsilon = \mathbb{R}^{\dim C_0}/\mathbb{R}^{\dim C_0 - 1} = \mathbb{R}. \tag{1.68}$$

We will prove the statement about ε in two steps

1. First inclusion

$$\text{Im } \partial_1 \subseteq \ker \varepsilon \tag{1.69}$$

follows from evaluation

$$\varepsilon(\partial_1 c) = \varepsilon \circ \partial_1 \sum c_{ij} e_{ij} = \varepsilon \left(\sum c_{ij}(e_j - e_i) \right) = \sum c_{ij}(\varepsilon(e_j) - \varepsilon(e_i)) = 0. \tag{1.70}$$

2. Second inclusion:

$$\ker \varepsilon \subseteq \text{Im } \partial_1. \tag{1.71}$$

For every

$$\ker \varepsilon \ni c = \sum c_i e_i, \quad \sum c_i = 0 \tag{1.72}$$

we can construct an "path" γ such that

$$\gamma = \sum c_i \gamma_{0i}, \quad \partial\gamma = \sum c_i e_i - e_0 \sum c_i = c. \tag{1.73}$$

The e_0 is some arbitrary reference point in Δ_M. The γ_{i0} is the path from vertex 0 to vertex i. An existence of γ_{0i} for every vertex i is guaranteed by the path-connectedness of M. The path γ_{0i} could be either a single 1d simplex or a linear combination of several 1d simplexex along the path from vertex 0 to vertex i

$$\gamma_{0i} = e_{0i_1} + e_{i_1 i_2} + ... + e_{ii_k}, \ \ \partial\gamma_{0i} = e_i - e_0. \tag{1.74}$$

The pair of inclusions

$$\text{Im } \partial_1 \subseteq \ker \varepsilon, \quad \ker \varepsilon \subseteq \text{Im } \partial_1 \tag{1.75}$$

lead to equality

$$\ker \varepsilon = \text{Im } \partial_1 \tag{1.76}$$

what concludes the proof of proposition.

1.7 Chain map

We introduced chain complexes as a models of topological spaces. Let us describe the chain complex version of the map between two topological spaces.

Definition: A morphism $f : C_\bullet \to D_\bullet$ between two chain complexes, also knows as the *chain map* is a series of maps $f_k : C_k \to D_k$ that commute with differentials d_C and d_D i.e.

$$f_{k-1} \circ \partial_k^C = \partial_k^D \circ f_k. \tag{1.77}$$

The diagrammatic representation complex-morphism is

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{k+1} & \xrightarrow{\partial_{k+1}^C} & C_k & \xrightarrow{\partial_k^C} & C_{k-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{k+1}} & & \downarrow{\scriptstyle f_k} & & \downarrow{\scriptstyle f_{k-1}} & & \\
\cdots & \longrightarrow & D_{k+1} & \xrightarrow[\partial_{k+1}^D]{} & D_k & \xrightarrow[\partial_k^D]{} & D_{k-1} & \longrightarrow & \cdots
\end{array}$$

with each square being the commutative diagram.

Proposition: The chain map $f : C_\bullet \to D_\bullet$ induces a homomorphism of homology groups

$$f_* : H_n(C) \to H_n(D). \tag{1.78}$$

Proof: By definition the chain map f maps cycle to cycle

$$f(\partial c) = \partial f(c) = 0, \ \ \forall c \in Z_n(C) \tag{1.79}$$

so it is well defined map from $Z_n(C)$ to $Z_n(D)$. Similarly f maps boundary to boundary, so it is well defined map of $B_n(C)$ to $B_n(D)$, therefore the map f maps $H_n(C)$ to $H_n(D)$.

Remark: Chain complexes with chain maps form a category of chain complexes $Ch_\bullet$

1.8 Homotopy

In our global definition of topological invariants we used the term "continuous deformations". In this section we will discuss a wide class of "continuous deformations" known as the *homotopy*. Let us start with the formal definition.

Definition: A *homotopy* between two continuous functions $f, g : X \to Y$ between topological spaces X and Y is a continuous function

$$H : X \times [0,1] \to Y, \quad H(x,0) = f(x), \quad H(x,1) = g(x). \tag{1.80}$$

We can think of the second argument as time, so the $H(t,x)$ describes a continuous deformation from f to g by the family of functions

$$H(t,x) = h_t(x) : X \to Y, \quad h_0 = f, \quad h_1 = g. \tag{1.81}$$

If for a pair of maps $f, g : X \to Y$ there exists a homotopy H then the two maps f and g are *homotopic* what we denote as

$$f \simeq g. \tag{1.82}$$

Example: Let $C \subset \mathbb{R}^n$ be a convex subset i.e. for all pairs $x, y \in C$ a straight line connecting x and y also belong to C. Let $f, g : [0,1] \to C$ are paths with the same endpoints then there is a homotopy given by

$$H : [0,1] \times [0,1] \to C \;:\; (s,t) \mapsto (1-t)f(s) + tg(s) \tag{1.83}$$

The image of $H(s,t)$ for fixed s is straight line between two points $f(s)$ and $g(s)$, hence such type of homotopy is called *linear homotopy* or *straight-line homotopy*.

Definition: Given two topological spaces X and Y, a *homotopy equivalence* between X and Y is a pair of continuous maps $f : X \to Y$ and $g : Y \to X$ such that

$$f \circ g \simeq \mathrm{id}_Y, \quad g \circ f \simeq Id_X \tag{1.84}$$

If such pair exists then we say that X is *homotopy equivalent* to Y and denote as

$$X \simeq Y. \tag{1.85}$$

Example: Any connected graph Γ is homotopy equivalent to the *rose with n petals* or bouquet of n circles with n being the number of loops on the graph

$$\Gamma \simeq \vee^n S^1. \tag{1.86}$$

Example: Euclidean space $\mathbb{R}^n$ is homotopy equivalent to a point i.e

$$\mathbb{R}^n \simeq \{0\}. \tag{1.87}$$

We can define embedding i and projection π maps

$$i : \{0\} \hookrightarrow \mathbb{R}^n, \quad \pi : \mathbb{R}^n \to \{0\} \tag{1.88}$$

with compositions

$$\pi \circ i = \mathrm{id}_0, \quad i \circ \pi = P_0 \tag{1.89}$$

with $P_0 : \mathbb{R}^n \to \mathbb{R}^n$ being projection to $\{0\}$ map. We can use the linear homotopy

$$H : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n \ : \ (x,t) \mapsto tP_0(x) + (1-t)\mathrm{id}_{\mathbb{R}^n} \tag{1.90}$$

to show that

$$P_0 \simeq \mathrm{id}_{\mathbb{R}^n} \tag{1.91}$$

hence

$$\pi \circ i = \mathrm{id}_0, \quad i \circ \pi \simeq \mathrm{id}_{\mathbb{R}^n} \quad \Longrightarrow \mathbb{R}^n \simeq \{0\}. \tag{1.92}$$

The homotopy equivalence $\mathbb{R}^n \simeq \{0\}$ is a example of the wider class of homotopy known as the *deformation retraction*.

Definition: a continuous map

$$F : X \times [0,1] \to X \tag{1.93}$$

is a *deformation retraction* of a space X onto subspace $A \subset X$ if

$$F(x,0) = x, \quad F(x,1) \in A, \quad F(a,t) = a, \quad \forall a \in A, \quad \forall x \in X. \tag{1.94}$$

In case such map exists subspace A is denoted as *deformation retract* of X.

Homotopy represents a wide class of continuous deformations for topological spaces, so we expect the topological invariants to be the same for homotopy equivalent spaces. We can use our simplicial models of the topological spaces to check this property. As the first step we need to describe the homotopy in terms of simplicial complexes. Let us observe that

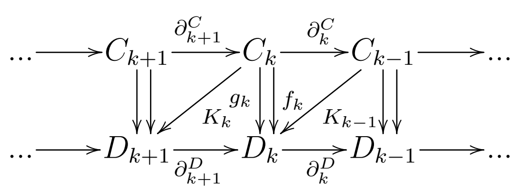
$$f(c) - g(c) = \partial(Hc) + H(\partial c) \tag{1.95}$$

what motivates the chain complex definition of homotopy.

Definition: Let $f, g : C_\bullet \to D_\bullet$ be two morphisms for chain complexes $C_\bullet$ and $D_\bullet$. The *chain homotopy* is a collection of maps $K_k : C_k \to D_{k+1}$ such that

$$f_k - g_k = \partial^D_{k+1} \circ K_k + K_{k-1} \circ \partial^C_k \tag{1.96}$$

The diagrammatic representation complex-morphism is



We can use the definition of the chain homotopy to describe the deformation retract for chain complexes.

Definition: Deformation retract of complex $C_\bullet$ to subcomplex $S_\bullet$ is a pair of maps

$$\pi : \; C_\bullet \to S_\bullet. \quad i : S_\bullet \hookrightarrow C_\bullet, \tag{1.97}$$

such that

$$\pi \circ i = \mathrm{id}_S, \quad i \circ \pi = \mathrm{id}_C + \partial \circ K + K \circ \partial \simeq \mathrm{id}_C. \tag{1.98}$$

with $K : C_\bullet \to C_{\bullet+1}$ being a *homotopy map*.

Proposition: Deformation retract preserves the homology. Let π, i, K is the deformation retract data from $C_\bullet$ to subcomplex $S_\bullet$ then

$$H(C_\bullet) = H(S_\bullet). \tag{1.99}$$

Proof: By construction being homotopic to the identity means that there exists a homotopy K such that

$$i \circ \pi = \mathrm{id}_C + \partial \circ K + K \circ \partial \tag{1.100}$$

The image of $H_k(C)$ under the the $\partial \circ K + K \circ \partial$ map is trivial as it maps closed chains to exact chains

$$(\partial \circ K + K \circ \partial)h = \partial K h = \partial(Kh) \simeq 0, \ \ \forall h \in H_k(C). \tag{1.101}$$

Example: Let us consider a simplest example of deformation retraction

$$\bullet_0 - \bullet_1 \ \ \mapsto \bullet_0 \tag{1.102}$$

The corresponding chain complexes and maps

$$\begin{array}{ccccccccc}
C_\bullet & & 0 & \longrightarrow & \mathbb{R} & \xrightarrow{\partial} & \mathbb{R}^2 & \longrightarrow & 0 \\
\downarrow{\scriptstyle \pi} & & & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \\
S_\bullet & & 0 & \longrightarrow & 0 & \xrightarrow[\partial]{} & \mathbb{R} & \longrightarrow & 0
\end{array}$$

The projection on 1-chains is trivial

$$\pi(e_{01}) = 0 \tag{1.103}$$

but we can use it to constrain the projection on 0-chains

$$\pi(e_0) = \alpha e_0, \ \ \pi(e_1) = \beta e_0. \tag{1.104}$$

Let us consider

$$0 = \partial \pi(e_{01}) = \pi(\partial e_{01}) = \pi(e_1 - e_0) = \pi(e_1) - \pi(e_0) \tag{1.105}$$

what leads to $\alpha = \beta$. The embedding map

$$i(e_0) = a e_0 + b e_1 \tag{1.106}$$

is an inverse of projection at $S^\bullet$ i.e

$$\mathrm{id}_S e_0 = e_0 = \pi \circ i(e_0) = \pi(ae_0 + be_1) = \alpha(a+b)e_0 \tag{1.107}$$

For simplicity let us make a symmetric choice $2a = 2b = \alpha = 1$ so that

$$\pi(e_0) = \pi(e_1) = e_0, \;\; \pi(e_{01}) = 0, \;\; i(e_0) = \frac{1}{2}(e_0 + e_1) \tag{1.108}$$

The homotopy

$$K(e_{01}) = 0, \;\; K(e_0) = \gamma_0 e_{01}, \;\; K(e_1) = \gamma_1 e_{01} \tag{1.109}$$

with γ_0 and γ_1 determined from

$$\begin{aligned} [i \circ \pi - \mathrm{id}_C](e_0) = \frac{1}{2}(e_0 + e_1) - e_0 = \frac{1}{2}(e_1 - e_0) \\ = [\partial \circ K + K \circ \partial](e_0) = \partial(\gamma_0 e_{01}) = \gamma_0(e_1 - e_0) \end{aligned} \tag{1.110}$$

and

$$\begin{aligned} [i \circ \pi - \mathrm{id}_C](e_1) = \frac{1}{2}(e_0 + e_1) - e_1 = \frac{1}{2}(e_0 - e_1) \\ = [\partial \circ K + K \circ \partial](e_1) = \partial(\gamma_1 e_{01}) = \gamma_1(e_1 - e_0) \end{aligned} \tag{1.111}$$

with final expression

$$K(e_0) = \frac{1}{2}e_{01}, \;\; K(e_1) = -\frac{1}{2}e_{01} \tag{1.112}$$

For completeness let us also check

$$\begin{aligned} [i \circ \pi - \mathrm{id}_C](e_{01}) = -e_{01} \\ = [\partial \circ K + K \circ \partial](e_{01}) = K(e_1 - e_0) = -\frac{1}{2}e_{01} - \left(\frac{1}{2}e_{01}\right) = -e_{01} \end{aligned} \tag{1.113}$$

An existence of homotopy K for the pair of simplicial complexes implies that

$$H_k^\Delta(I) = H(\Delta_I) = H(C) = H(S) = H(\Delta_{pt}) = H_k^\Delta(pt) = \begin{cases} \mathbb{R}, & k = 0; \\ 0, & k \neq 0. \end{cases} \tag{1.114}$$