Supersymmetric Quantum Mechanics and Morse theory

Vyacheslav Lysov

Okinawa Institute for Science and Technology

Lectures 21-22: 1d Supersymmetry

Contents

1 1d Supersymmetry

1.1 Harmonic oscillator in holomorphic polarization

Let us consider Harmonic oscillator, a particle of unit mass on a real line $X = \mathbb{R}$ in quadratic potential

$$L = \frac{\dot{x}^2}{2} - \frac{\omega^2 x^2}{2} \tag{1.1}$$

The hamiltonian description of the same particle consists of phase space a 2d manifold

$$\mathcal{M} = T^*X = T^*\mathbb{R} = \mathbb{R}^2_{px} \tag{1.2}$$

with symplectic structure

$$\omega = dp \wedge dx \tag{1.3}$$

and Hamiltonian

$$H = \frac{p^2}{2} + \frac{\omega^2 x^2}{2}. \tag{1.4}$$

We can describe the same system in complex coordinates

$$z = \frac{\omega x + ip}{\sqrt{2\omega}}, \quad \bar{z} = \frac{\omega x - ip}{\sqrt{2\omega}}. \tag{1.5}$$

The symplectic structure in complex coordinates

$$\omega = dp \wedge dx = i d\bar{z} \wedge dz \tag{1.6}$$

while the Hamiltonian

$$H = \omega z \bar{z}. \tag{1.7}$$

The Poisson bracket in complex coordinates is generated by

$$\{z, \bar{z}\}_{pb} = \iota_{X_{\bar{z}}} dz = -i. \tag{1.8}$$

Canonical quantization tells us that the corresponding operators obey commutation relation

$$[\hat{\bar{z}}, \hat{z}] = -i\hbar \widehat{\{\bar{z}, z\}} = \hbar. \tag{1.9}$$

Let us notice that the classical observables z and $\bar{z}$ are complex-valued functions, hence the corresponding operators not need to be hermitian. However the real and imaginary parts of z are real-valued functions, so there should be corresponding Hermitian operators. We can express real and imaginary parts

$$\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z}), \;\; \mathrm{Im}(z) = \frac{1}{2i}(z - \bar{z}), \tag{1.10}$$

so the operators $\hat{z}$ and $\hat{\bar{z}}$ should be hermitian conjuations of each other

$$\hat{z}^\dagger = \hat{\bar{z}}, \;\; \hat{\bar{z}}^\dagger = \hat{z}. \tag{1.11}$$

We can rewrite the commutation (1.9) relations in terms of $\hat{z}$ and $\hat{z}^\dagger$

$$[\hat{\bar{z}}, \hat{\bar{z}}^\dagger] = \hbar, \tag{1.12}$$

which is a familiar commutation relation for creation-annihilation operators of Harmonic oscillator

$$\hat{\bar{z}} = \sqrt{\hbar}\, a, \quad \hat{z} = \sqrt{\hbar}\, a^\dagger. \tag{1.13}$$

The Hamiltonian for Harmonic oscillator in terms of $a, a^\dagger$ is of the form

$$H = \frac{\hat{p}^2}{2} + \frac{\omega^2 \hat{x}^2}{2} = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right) \tag{1.14}$$

while the spectrum of the Hamiltonian we can describe using Fock representation. The ground state $|0\rangle$ obeys

$$a|0\rangle = \left(\frac{\omega \hat{x} + i\hat{p}}{\sqrt{2\hbar\omega}} \right) |0\rangle = 0. \tag{1.15}$$

The wavefuction for ground state in position basis

$$\psi_0 = \langle x|0\rangle \tag{1.16}$$

is a solution linear differential equation

$$\left(\frac{\omega x + \hbar\partial_x}{\sqrt{2\hbar\omega}}\right)\psi_0(x) = 0. \tag{1.17}$$

The normalized solution is

$$\psi_0(x) = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{\omega x^2}{2\hbar}}. \tag{1.18}$$

The states in a Hilbert space of our system can be constructed by action of raising operator $a^\dagger$ on ground state. The normalized version of the states

$$\langle n|m\rangle = \delta_{n,m}, \quad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle \tag{1.19}$$

form a complete basis in Hilbert space $\mathcal{H}$

$$1_{\mathcal{H}} = \sum_{n=0}^{\infty}|n\rangle\langle n|. \tag{1.20}$$

The states $|n\rangle$ are eigenstates of hamiltonian

$$\hat{H}(a^\dagger)^n|0\rangle = \hbar\omega\left(n + \frac{1}{2}\right)(a^\dagger)^n|0\rangle. \tag{1.21}$$

Thought the annihilation operator a is not Hermitian, its eigenfunctions form the *over-complete basis* for $\mathcal{H}$. The eigenfunctions of a are known in physics literature as *coherent states*

$$|z\rangle = \sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\hbar^n n!}}|n\rangle = \sum_{n=0}^{\infty}\frac{z^n(a^\dagger)^n}{\hbar^{n/2}n!}|0\rangle = e^{\frac{za^\dagger}{\sqrt{\hbar}}}|0\rangle. \tag{1.22}$$

The overcompleteness results into states being not orthogonal

$$\langle z|w\rangle = e^{\frac{\bar{z}w}{\hbar}} \tag{1.23}$$

thought the completeness is still there

$$1_{\mathcal{H}} = \frac{1}{\pi\hbar}\int_{\mathbb{C}} d^2z e^{-\frac{z\bar{z}}{\hbar}}|z\rangle\langle z|. \tag{1.24}$$

We can check the relation above using the complete set of energy eigenstates $|n\rangle$

$$
\begin{aligned}
1_{\mathcal{H}} &= \frac{1}{\pi\hbar}\int_{\mathbb{C}} d^2z e^{-\frac{z\bar{z}}{\hbar}}|z\rangle\langle z| = \frac{1}{\pi\hbar}\int_{\mathbb{C}} d^2z e^{-z\bar{z}}\sum_{m,n}\frac{z^n\bar{z}^m}{\sqrt{\hbar^{n+m}m!n!}}|n\rangle\langle m| \\
&= \frac{1}{\pi\hbar}\sum_{m,n}\int_0^{2\pi} e^{i(n-m)\phi}d\phi\int_0^\infty rdr\ e^{-\frac{r^2}{\hbar}}\frac{r^{n+m}}{\sqrt{\hbar^{n+m}m!n!}}|n\rangle\langle m| \\
&= \frac{1}{\pi}\sum_{mn}2\pi\delta_{m,n}\int_0^\infty rdr\ e^{-r^2}\frac{r^{m+n}}{\sqrt{m!n!}}|n\rangle\langle m| \\
&= \sum_n\int_0^\infty dr^2\ e^{-r^2}\frac{r^{2n}}{n!}|n\rangle\langle n| = \sum_n|n\rangle\langle n| = 1_{\mathcal{H}}.
\end{aligned}
\tag{1.25}
$$

The state $|\Psi\rangle$ in coherent state basis is described by the wavefunction

$$
\Psi(z) = \langle z|\Psi\rangle, \quad \hat{\bar{z}}|z\rangle = z|z\rangle, \quad \langle z|\hat{z} = \langle z|\bar{z} \tag{1.26}
$$

which is by construction an eigenfunction of $\hat{z}$ i.e.

$$
\hat{z}\Psi(z) = z\Psi(z) \tag{1.27}
$$

We can use our coherent state discussion to formulate the quantization of $\mathbb{C}$ in *holomorphic polarization.* The Hilbert space is the space of holomorphic functions

$$
\mathcal{H}_J = C^\infty_{hol}(\mathbb{C}) \tag{1.28}
$$

with pairing

$$
\langle\Phi|\Psi\rangle = \frac{1}{\pi\hbar}\int_{\mathbb{C}} d^2z e^{-\frac{z\bar{z}}{\hbar}}\langle\Phi|z\rangle\langle z|\Psi\rangle = \langle\Phi,\Psi\rangle = \frac{1}{2\pi i\hbar}\int_{\mathbb{C}} dzd\bar{z}\ e^{-\frac{z\bar{z}}{\hbar}}\overline{\Phi(z)}\Psi(z). \tag{1.29}
$$

The operators $\hat{z}$ and $\hat{\bar{z}}$ in holomorphic polarization become derivative and multiplication

$$
\hat{z} = z\cdot, \quad \hat{\bar{z}} = \hbar\partial_z \tag{1.30}
$$

The hermitian conjugate operator $\hat{\bar{z}}^\dagger$ with respect to the paring on Hilbert

$$\begin{aligned}\langle \hat{\bar{z}}^\dagger \Phi, \Psi\rangle = \langle \Phi, \hat{\bar{z}}\Psi\rangle &= \frac{1}{2\pi i\hbar}\int_{\mathbb{C}} dz d\bar{z}\; e^{-\frac{z\bar{z}}{\hbar}}\overline{\Phi(z)}\hbar\partial_z\Psi(z) \\ &= -\frac{1}{2\pi i\hbar}\int_{\mathbb{C}} dz d\bar{z}\; \hbar\partial_z(e^{-\frac{z\bar{z}}{\hbar}})\overline{\Phi(z)}\Psi(z) = \frac{1}{2\pi i\hbar}\int_{\mathbb{C}} dz d\bar{z}\; e^{-\frac{z\bar{z}}{\hbar}}\,\overline{z\Phi(z)}\Psi(z) \\ &= \langle \hat{z}\Phi, \Psi\rangle\end{aligned} \tag{1.31}$$

matches with the $\hat{\bar{z}}$ operator. The quantum Hamiltonian

$$\hat{H} = \omega\widehat{z\bar{z}} \tag{1.32}$$

is defined up to a constant due to possible choice of ordering for $\hat{z}$ and $\hat{\bar{z}}$. The familiar harmonic oscillator spectrum corresponds to symmetric ordering

$$\hat{H} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) = \omega\left(\hat{z}\hat{\bar{z}} + \hat{\bar{z}}\hat{z}\right) = \hbar\omega\left(z\partial_z + \frac{1}{2}\right) \tag{1.33}$$

Let us explain this choice from holomorphic quantization perspective. Our original intention was to quantize the $\mathbb{R}^2 = T^*\mathbb{R}$ which we promoted to $\mathbb{C}$ by picking up a complex structure J. There is no canonical choice of J so we expect the physical data (spectrum of H) being J-independent. The change in complex structure acts a linear transformation on $z, \bar{z}$ and on corresponding operators. The Hamiltonian in symmetric ordering is invariant under such action.

Let us also derive the trace formula in coherent state basis, starting with trace in energy eigenstates representation

$$\begin{aligned}\mathrm{Tr}\mathcal{O} = \sum_{n=0}^{\infty}\langle n|\mathcal{O}|n\rangle &= \frac{1}{\pi^2\hbar^2}\sum_{n=0}^{\infty}\int_{\mathbb{C}} d^2w e^{-\frac{w\bar{w}}{\hbar}}\int_{\mathbb{C}} d^2z e^{-\frac{z\bar{z}}{\hbar}}\langle n|w\rangle\langle w|\mathcal{O}|z\rangle\langle z|n\rangle \\ &= \frac{1}{\pi^2\hbar^2}\int_{\mathbb{C}} d^2w e^{-\frac{w\bar{w}}{\hbar}}\int_{\mathbb{C}} d^2z e^{-\frac{z\bar{z}}{\hbar}}\langle w|\mathcal{O}|z\rangle\sum_{n=0}^{\infty}\langle z|n\rangle\langle n|w\rangle \\ &= \frac{1}{\pi^2\hbar^2}\int_{\mathbb{C}} d^2w e^{-\frac{w\bar{w}}{\hbar}}\int_{\mathbb{C}} d^2z e^{-\frac{z\bar{z}}{\hbar}}\langle w|\mathcal{O}|z\rangle\langle z|w\rangle \\ &= \frac{1}{\pi\hbar}\int_{\mathbb{C}} d^2w e^{-\frac{w\bar{w}}{\hbar}}\langle w|\mathcal{O}|w\rangle\end{aligned} \tag{1.34}$$

1.2 Grassmann harmonic oscillator

We can generalize our analysis for harmonic oscillator to fermionic harmonic oscillator. We replace complex coordinates $z, \bar{z}$ by complex Grassmann coordinates ψ and $\bar{\psi}$. The phase space is Grasmann manifold $\mathbb{R}^{0|2} \simeq \mathbb{C}^{0|1}$ with symplectic structure

$$\omega = i d\bar{\psi} \wedge d\psi \tag{1.35}$$

The Hamiltonian of fermionic harmonic oscillator

$$H = \omega_F \psi \bar{\psi} \tag{1.36}$$

The (graded) Poisson brackets on coordinate functions

$$\{\bar{\psi}, \psi\}_{pb} = i, \quad \{\psi, \psi\}_{pb} = \{\bar{\psi}, \bar{\psi}\}_{pb} = 0 \tag{1.37}$$

in canonical quantization become graded commutators of operators

$$\{\hat{\bar{\psi}}, \hat{\psi}\} = -i\hbar \cdot \widehat{\{\bar{\psi}, \psi\}}_{pb} = \hbar \cdot 1_H \tag{1.38}$$

By construction Grassmann variables ψ and $\bar{\psi}$ are complex conjugate so the the corresponding operators are Hermitian conjugate

$$\hat{\psi}^\dagger = \hat{\bar{\psi}}. \tag{1.39}$$

We can realize the $\hat{\psi}$ operators as linear operators on a Hilbert spaces in three different ways

1. **Fermionic creation-annihilation operators**. We can generalize operators a and $a^\dagger$ from harmonic oscillator to Grassmann-odd operators with similar algebra

$$\{b, b^\dagger\} = 1, \quad b^2 = (b^\dagger)^2 = 0. \tag{1.40}$$

   We can represent the $\hat{\psi}$ and $\hat{\bar{\psi}}$ in terms of b and $b^\dagger$ via

$$\hat{\psi} = \sqrt{\hbar} b, \quad \hat{\bar{\psi}} = \sqrt{\hbar} b^\dagger \tag{1.41}$$

The ground state $|0\rangle$ is defined as state annihilated by lowering operator b

$$b|0\rangle = 0, \tag{1.42}$$

The Hilbert space is Fock space created by raising operator $b^\dagger$ acting on the ground state $|0\rangle$

$$\mathcal{H} = \mathbb{C}\langle|0\rangle, b^\dagger|0\rangle\rangle \simeq \mathbb{C}^2. \tag{1.43}$$

The action limited to single creation operator since it is niltoptent. The Hamiltonian

$$\hat{H} = \omega\widehat{\psi\bar{\psi}} = \hbar\omega[b^\dagger, b] = \hbar\omega\left(b^\dagger b - \frac{1}{2}\right) \tag{1.44}$$

has two energy levels

$$\begin{aligned}\hat{H}|0\rangle &= -\frac{1}{2}\hbar\omega|0\rangle \\ \hat{H}|1\rangle &= \hat{H}b^\dagger|0\rangle = \frac{1}{2}\hbar\omega b^\dagger|0\rangle = \frac{1}{2}\hbar\omega|1\rangle.\end{aligned} \tag{1.45}$$

2. **Differential operators on holomorphic functions** The Hilbert space is the space of hoomorphic functions

$$\mathcal{H} = \mathbb{C}_{hol}[\mathbb{C}^{0|1}] \simeq \mathbb{C}^2 \tag{1.46}$$

with inner product

$$\langle\Phi, \Psi\rangle = \int d\psi d\bar{\psi}\ \ e^{\frac{\psi\bar{\psi}}{\hbar}}\Psi(\psi)\overline{\Phi(\psi)} \tag{1.47}$$

The operators realized as

$$\hat{\psi} = \psi\cdot, \quad \hat{\bar{\psi}} = \hbar\frac{\partial}{\partial\psi} \tag{1.48}$$

The Hamiltonian is

$$\hat{H} = \hbar\omega_F\left(\psi\partial_\psi - \frac{1}{2}\right) \tag{1.49}$$

The eigenstates $|\psi\rangle$ form the resolution of the identity

$$1_{\mathcal{H}} = \int d\psi d\bar{\psi}e^{\psi\bar{\psi}}\ \ |\psi\rangle\langle\bar{\psi}| \tag{1.50}$$

3. **2×2 matrices** We can use the fact that Hilbert space is 2-dimensional to describe states as vectors

$$\mathbb{C}[\psi] \simeq \mathbb{C}^2, \quad \Psi = \begin{pmatrix} \Psi_0 \\ \Psi_1 \end{pmatrix} \tag{1.51}$$

and represent linear operators on it as a 2×2 matrices

$$\hat{\psi} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \hat{\bar{\psi}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.52}$$

The hermitian structure is the multiplication of two vectors

$$\langle \Phi, \Psi \rangle = \overline{\Phi_0}\Psi_0 + \overline{\Phi_1}\Psi_1 = \Phi^\dagger \Psi \tag{1.53}$$

The Hamiltonian for Harmonic oscillator is

$$\hat{H} = \frac{1}{2}\hbar\omega_F \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.54}$$

Each of three representation for quantum fermionic oscillator has useful applications. We will use creation-annihilation representation to describe the supersymmetric harmonic oscillator as it makes supersymmetry very explicit. We will use the coherent state representation to relate the states in supersymmetric Hilbert space to differential forms. The matrix representation allows us to avoid using Grassmann variables so we can use it to recognize the hidden supersymmetry in purely bosonic systems like electron in magnetic field.

1.3 $N = 2$ supersymmetric harmonic oscillator

Let us consider a pair of quantum harmonic oscillators: the first one is bosonic with frequency ω_B while the second one is fermionic, with frequency ω_F. The Hilbert space of this system, being the tensor product of individual Hilbert spaces

$$\mathcal{H} = \mathcal{H}_B \otimes \mathcal{H}_F \tag{1.55}$$

can be realized as lowest weight representation of the algebra

$$\begin{gathered} [a, a^\dagger] = \{b, b^\dagger\} = 1, \quad b^2 = (b^\dagger)^2 = 0 \\ [a, b] = [a, b^\dagger] = [a^\dagger, b] = [a^\dagger, b^\dagger] = 0. \end{gathered} \tag{1.56}$$

The lowest weight state $|0\rangle$ is annihilated by both types of lowering operators

$$a|0\rangle = b|0\rangle = 0. \tag{1.57}$$

We can label the other states in the Hilbert space by pair of integers n_B and n_F so that

$$|n_B, n_F\rangle = \frac{(a^\dagger)^{n_B}}{\sqrt{n_B!}}(b^\dagger)^{n_F}|0\rangle. \tag{1.58}$$

The Grassmann-odd nature of the fermionic creation operator limits the range of possible values for n_B to be 0 or 1. The Hamiltonian for SUSY oscillator

$$H = \hbar\omega_F\left(b^\dagger b - \frac{1}{2}\right) + \hbar\omega_B\left(a^\dagger a + \frac{1}{2}\right) \tag{1.59}$$

is diagonal in basis (1.58) with energies

$$E(n_B, n_F) = E(|n_B, n_F\rangle) = \hbar\omega_B\left(n_B + \frac{1}{2}\right) + \hbar\omega_F\left(n_F - \frac{1}{2}\right) \tag{1.60}$$

Supersymmetric oscillator has equal frequencies $\omega_B = \omega_F = \omega$ so the spectrum becomes

$$E(n_B, n_F) = \hbar\omega(n_B + n_F), \quad n_F = 0, 1 \quad n_B = 0, 1, 2, .. \tag{1.61}$$

Let us make several observations about the spectrum

- All states have positive energies

$$E(n_B, n_F) = \hbar\omega(n_B + n_F) \geq 0. \tag{1.62}$$

- There is single lowest energy state with $E = 0$

$$|0, 0\rangle = |0\rangle, \quad E(|0\rangle) = 0. \tag{1.63}$$

- Energy levels above zero energy come in pairs

$$E(n_B, 1) = E(n_B + 1, 0). \tag{1.64}$$

- The two states with the same energy have different parity - one is even, in physics terminology bosonic state $|n_B + 1, 0\rangle$ while another $|n_B, 1\rangle$ is odd, fermionic in physics terminology. We can introduce the *fermionic number operator* defined via

$$(-)^F|n_B, n_F\rangle = (-1)^{n_F}|n_B, n_F\rangle, \tag{1.65}$$

so we can refer the states with eigenvalue $+1$ as bosonic, while the ones with eigenvalue -1 as fermionic states.

Let us now try to formalize this observations. We can introduce a symmetry operator Q that relates the states with the same energy

$$Q|n_B, n_F\rangle \propto |n_B + 1, n_F - 1\rangle \tag{1.66}$$

and express it in terms of oscillators

$$Q = \sqrt{\hbar\omega} b^\dagger a. \tag{1.67}$$

By construction the symmetry $\hat{Q}$ has certain properties

- Q is nilpotent
$$Q^2 = \hbar\omega(b^\dagger a)^2 = \hbar\omega(b^\dagger)^2 a^2 = 0 \tag{1.68}$$

- Q maps bosonic to fermionic states i.e it anticommutes with $(-)^F$
$$(-1)^F Q = -Q(-1)^F \tag{1.69}$$

- The Hamiltonian $\hat{H}$ can be expressed in terms of Q
$$\begin{aligned} \{Q, Q^\dagger\} &= \hbar\omega(b^\dagger a a^\dagger b + a^\dagger b b^\dagger a) = \hbar\omega(a a^\dagger b^\dagger b + a^\dagger a b b^\dagger) \\ &= \hbar\omega((1 + a^\dagger a) b^\dagger b + a^\dagger a(1 - b^\dagger b)) = \hbar\omega(b^\dagger b + a^\dagger a) = \hat{H} \end{aligned} \tag{1.70}$$

The operator Q, which obeys above properties is called *supersymmetry generator*. In the next section we will show that the spectrum of quantum theory with supersymmetry is very similar to the supersymmetric harmonic oscillator.

1.4 $d = 1$ $N = 2$ SUSY algebra

The algebra

$$\begin{aligned} \hat{H} &= \{Q, Q^\dagger\}, \\ \{Q, Q\} &= \{Q^\dagger, Q^\dagger\} = 0 \end{aligned} \tag{1.71}$$

is known as the $d = 1$ $N = 2$ SUSY algebra. The remaining commutation relations

$$[Q, H] = [Q^\dagger, H] = 0 \tag{1.72}$$

follow from (1.71) via graded Jacobi identities

$$\{A, \{B, C]] + (-1)^{|A|(|B|+|C|)}\{B, \{C, A]] + (-1)^{|B|(|C|+|A|)}\{C, \{A, B]] = 0. \tag{1.73}$$

$$[Q, H] = [Q, \{Q, Q^\dagger\}] = -\{Q, \{Q^\dagger, Q]] - \{Q^\dagger, \{Q, Q]] = 0 \tag{1.74}$$

There is an alternative form of the $d = 1$ SUSY algebra

$$\{Q_i, Q_j\} = 2\hat{H}\delta_{ij}, \quad i, j = 1..N, \tag{1.75}$$

valid for any number of supercharges N. The two algebras are the related via

$$Q_1 = Q + Q^\dagger, \quad Q_2 = i(Q - Q^\dagger) \tag{1.76}$$

We can show that the nice properties of the spectrum of supersymmetric harmonic oscillator generalize to arbitrary quantum mechanical theory in presence of SUSY algebra

- Energy levels above zero energy are degenerate. Indeed given a state $|\Psi\rangle$ with energy E we can construct another state $Q|\Psi\rangle$ with the same energy

$$\hat{H}Q|\Psi\rangle = Q\hat{H}|\Psi\rangle = QE|\Psi\rangle = EQ|\Psi\rangle \tag{1.77}$$

- states $|\Psi\rangle$ and $Q|\Psi\rangle$ have different fermionic numbers

$$(-1)^F Q|\Psi\rangle = -Q(-1)^F|\Psi\rangle \tag{1.78}$$

- All energy states have positive energies

$$E(|\Psi\rangle) = \langle\Psi|\hat{H}|\Psi\rangle = \langle\Psi|(QQ^\dagger + Q^\dagger Q)|\Psi\rangle = ||Q|\Psi\rangle||^2 + ||Q^\dagger|\Psi\rangle||^2 \geq 0 \tag{1.79}$$

- The ground state has $E = 0$ and is a single state

$$E(|\Psi\rangle) = 0 \ \Leftrightarrow Q|\Psi\rangle = Q^\dagger|\Psi\rangle = 0 \tag{1.80}$$

- We can define *supersymmetric index* also known as the *Witten index* in physics literature

$$I_W(\beta) = \mathrm{Tr}_{\mathcal{H}}(-1)^F e^{-\beta\hat{H}} = \mathrm{Tr}_{\mathcal{H}_0}(-1)^F = n_0^B - n_0^F = 1 \tag{1.81}$$

  which can be used to probe the supersymmetric ground states of the theory. In particular in a limit

$$\lim_{\beta\to\infty} I_W(\beta) = \mathrm{Tr}_{\mathcal{H}}(-1)^F \lim_{\beta\to\infty} e^{-\beta\hat{H}} = \mathrm{Tr}_{\mathcal{H}_0}(-1)^F = n_0^B - n_0^F \tag{1.82}$$

  We can show that the Witten index is independent of β

$$\begin{aligned}\partial_\beta I_W(\beta) &= -\mathrm{Tr}(-)^F\hat{H}e^{-\beta\hat{H}} = -\mathrm{Tr}(-)^F(QQ^\dagger + Q^\dagger Q)e^{-\beta\hat{H}} = \\ &= \mathrm{Tr}(Q(-)^F Q^\dagger + Q^\dagger(-)^F Q)e^{-\beta\hat{H}} = \mathrm{Tr}((-)^F Q^\dagger e^{-\beta\hat{H}}Q + (-)^F Q e^{-\beta\hat{H}}Q^\dagger) \\ &= \mathrm{Tr}((-)^F Q^\dagger Q e^{-\beta\hat{H}} + (-)^F QQ^\dagger e^{-\beta\hat{H}}) = \mathrm{Tr}(-)^F\hat{H}e^{-\beta\hat{H}} \\ &= -\partial_\beta I_W(\beta) = 0\end{aligned} \tag{1.83}$$

  where we used the cyclicity of the trace and commutators between $Q, \hat{H}$ and $(-)^F$.

1.5 Action principle for supersymmetric oscillator

The phase space for the supersymmetric oscillator

$$\mathcal{M} = \mathcal{M}_B \times \mathcal{M}_F = \mathbb{R}^{2|0} \times \mathbb{R}^{0|2} = \mathbb{R}^{2|2} = \mathbb{C}^{1|1} \tag{1.84}$$

with symplectic structure

$$\omega = id\bar{z} \wedge dz + id\bar{\psi} \wedge d\psi \tag{1.85}$$

The Hamiltonian for the system

$$H = \omega_F \psi\bar{\psi} + \omega_B z\bar{z} \tag{1.86}$$

We discussed nice properties of the SUSY algebra and a particular realization of the SUSY algebra for supersymmetric harmonic oscillator in Fock basis. In later section we will see that the most examples of SUSY QM naturally came in the form of minimal action principle, so let us trace back the SUSY algebra to the level symmetry of the action.

The symplectic structure SUSY phase space is responsible for the derivative term in the Hamiltonian action. The derivative term is the integral over world-line of the pullback of the one form θ such that

$$d\theta = \omega \tag{1.87}$$

Such one forms are not defined uniquely, while we can use the polarization data to fix the θ. In our case we want action to be real function, i.e. invariant under complex conjugation. The corresponding choice of 1 form θ lead to the kinetic term in the cation being

$$S_{kin} = \int_I \gamma^*\theta = \frac{i}{2}\int dt \quad \left(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi + \bar{z}\dot{z} - \dot{\bar{z}}z\right) \tag{1.88}$$

Since ψ are not commuting variables we need to use hermitian conjugation instead on the usual complex conjugation

$$S^\dagger_{kin} = -\frac{i}{2}\int dt \quad \left(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi + \bar{z}\dot{z} - \dot{\bar{z}}z\right)^\dagger = -\frac{i}{2}\int dt \quad \left(\dot{\bar{\psi}}\psi - \bar{\psi}\dot{\psi} + \dot{\bar{z}}z - \bar{z}\dot{z}\right) = S_{kin} \tag{1.89}$$

The full action includes both symplectic structure and the Hamiltonian

$$S = \int_I \gamma^*\theta - \int H dt = \int dt \left(\frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi + \bar{z}\dot{z} - \dot{\bar{z}}z) - \omega\bar{\psi}\psi - \omega\bar{z}z\right). \tag{1.90}$$

The variables ψ and z enter the action in symmetric way, so it is not surprising that the action is invariant under the rotations of $\psi, \bar{\psi}$ into $z, \bar{z}$. However z and ψ have different parities to the parameters $\epsilon, \bar{\epsilon}$ of rotation should be Grassman numbers

$$\begin{aligned}
\delta z &= -\bar{\epsilon}\psi, \\
\delta \bar{z} &= \epsilon\bar{\psi} \\
\delta \psi &= \epsilon z \\
\delta \bar{\psi} &= \bar{\epsilon}\bar{z}
\end{aligned} \tag{1.91}$$

Let us compare these two symmetries with the supersymmetry in quantum theory. We can use Poisson brackets

$$\{\bar{\psi}, \psi\}_{pb} = i, \quad \{\bar{z}, z\}_{pb} = i \tag{1.92}$$

to describe symmetry transformations as poisson brackets. For arbitrary function F on phase space

$$\begin{aligned} \delta F &= \{F, \bar{\epsilon}\mathcal{Q} - \epsilon\bar{\mathcal{Q}}\}_{pb} \\ \mathcal{Q} &= i\sqrt{2\omega}\psi\bar{z}, \quad \bar{\mathcal{Q}} = -i\sqrt{2\omega}\bar{\psi}z. \end{aligned} \tag{1.93}$$

We choose to introduce extra factors of $\sqrt{\omega}$ so $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ form standard $d = 1$ $N = 2$ SUSY algebra via

$$\begin{aligned} -i\{\mathcal{Q}, \bar{\mathcal{Q}}\}_{PB} &= 2(\omega\bar{\psi}\psi + \omega z\bar{z}) = 2H \\ \{\mathcal{Q}, \mathcal{Q}\}_{PB} &= \{\bar{\mathcal{Q}}, \bar{\mathcal{Q}}\}_{PB} = 0 \end{aligned} \tag{1.94}$$

The canonical quantization of $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ using ocillator representation

$$\hat{\mathcal{Q}} = -i\sqrt{2\omega}\hat{\psi}\hat{\bar{z}} \propto \sqrt{\omega\hbar}b^{\dagger}a = Q \tag{1.95}$$

show the perfect match between Grassmann symmetry of the action and $d = 1$ $N = 2$ SUSY acting on Hilbert space of supersymmetric harmonic oscillator.

The supersymmetry transformations in complex coordinates $z, \bar{z}, \psi, \bar{\psi}$ are quite easy to spot from the action, while it is relatively harder to observe the same symmetry in p, x coordinates. We can use an explicit change of variables (1.5) to describe the action and symmetries for supersymmetric harmonic oscillator in $p, x, \psi, \bar{\psi}$ coordinates

$$S = \int dt \left(p\dot{x} + \frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - \omega\bar{\psi}\psi - \frac{p^2}{2} - \frac{\omega^2 x^2}{2} \right) \tag{1.96}$$

The SUSY transfromations in these coordinates

$$\begin{aligned}
\delta x &= \frac{\delta z + \delta \bar{z}}{\sqrt{2\omega}} = \frac{\epsilon\bar{\psi} - \bar{\epsilon}\psi}{\sqrt{2\omega}} \\
\delta p &= -i\sqrt{\omega}\frac{\delta z - \delta \bar{z}}{\sqrt{2}} = -i\sqrt{\frac{\omega}{2}}(\bar{\epsilon}\psi + \epsilon\bar{\psi}) \\
\delta \psi &= \frac{\epsilon(\omega x + ip)}{\sqrt{2\omega}} \\
\delta \bar{\psi} &= \frac{\bar{\epsilon}(\omega x - ip)}{\sqrt{2\omega}}
\end{aligned} \tag{1.97}$$

We can perform the Legendre transform from p to $\dot{x}$ variables to move from Hamiltonian to Lagrangian description

$$\dot{x} = \frac{\partial H}{\partial p} = p, \quad L = p\dot{x} - H \tag{1.98}$$

The Lagrangian action

$$S = \int dt L = \int dt \left(\frac{1}{2}\dot{x}^2 + \frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - \omega\bar{\psi}\psi - \frac{\omega^2 x^2}{2} \right) \tag{1.99}$$

is invariant under the SUSY transformations

$$\begin{aligned}
\delta x &= \epsilon\bar{\psi} - \bar{\epsilon}\psi \\
\delta \psi &= \epsilon(i\dot{x} + \omega x) \\
\delta \bar{\psi} &= \bar{\epsilon}(-i\dot{x} + \omega x)
\end{aligned} \tag{1.100}$$

Remark: The supersymmetry transformations above form $d = 1$ $N = 2$ algebra up to equations of motions, hence the supersymmetry is the on-shell supersymmetry. In order to verify this claim let us introduce the generators of SUSY transfromations acting of functions of time

$$\mathbf{Q} = \int dt \left[-\psi(t)\frac{\delta}{\delta x(t)} + (-i\dot{x}(t) + \omega x(t))\frac{\delta}{\delta \bar{\psi}(t)} \right] \tag{1.101}$$

$$\bar{\mathbf{Q}} = \int dt \left[\bar{\psi}(t)\frac{\delta}{\delta x(t)} + (i\dot{x}(t) + \omega x(t))\frac{\delta}{\delta \psi(t)} \right] \tag{1.102}$$

The algebra of generators can be easily derived

$$
\begin{aligned}
\mathbf{Q}^2 x &= \bar{\mathbf{Q}}^2 x = 0 \\
\mathbf{Q}^2 \psi &= \bar{\mathbf{Q}}^2 \psi \approx 0 \\
\mathbf{Q}^2 \bar{\psi} &= \bar{\mathbf{Q}}^2 \bar{\psi} \approx 0 \\
\{\mathbf{Q}, \bar{\mathbf{Q}}\} x(t) &= -i\dot{x} - \omega x - i\dot{x} + \omega x = -2i\dot{x}(t) \\
\{\mathbf{Q}, \bar{\mathbf{Q}}\} \psi(t) &= -i\dot{\psi} - \omega\psi \approx -2i\dot{\psi} \\
\{\mathbf{Q}, \bar{\mathbf{Q}}\} \bar{\psi}(t) &= -i\dot{\bar{\psi}} + \omega\bar{\psi} \approx -2i\dot{\bar{\psi}}
\end{aligned}
\tag{1.103}
$$

The $\approx$ sign stands for using the equations of motion

$$
\begin{aligned}
\frac{\delta S}{\delta \bar{\psi}(t)} &= i\dot{\psi}(t) - \omega\psi(t) = 0 \\
\frac{\delta S}{\delta \psi(t)} &= i\dot{\bar{\psi}}(t) + \omega\bar{\psi}(t) = 0
\end{aligned}
\tag{1.104}
$$

1.6 Differential forms from $1d$ SUSY QM

Let us consider n identical copies of free 1d supersymmetric particle, i.e theory with action

$$
S = \frac{1}{2} \sum_{j=1}^{n} \int dt \; (i\bar{\psi}^j \dot{\psi}^j - i\dot{\bar{\psi}}^j \psi^j + \dot{x}^j \dot{x}^j)
\tag{1.105}
$$

We can interpret the action above as the action for free supersymmetric particle moving on $\mathbb{R}^n$. The Hamiltonian description of the same system

$$
S = \frac{1}{2} \sum_{j=1}^{n} \int dt \; (i\bar{\psi}^j \dot{\psi}^j - i\dot{\bar{\psi}}^j \psi^j + 2p_j \dot{x}^j - p_j^2)
\tag{1.106}
$$

invariant under the diagonal, i.e. having identical parameters $\epsilon, \bar{\epsilon}$ for all values of j, supersymmetry transformations

$$
\begin{aligned}
\delta x^j &= \epsilon \bar{\psi}^j - \bar{\epsilon} \psi^j \\
\delta \psi^j &= i\epsilon p_j \\
\delta \bar{\psi}^j &= -i\bar{\epsilon} p_j \\
\delta p_j &= 0
\end{aligned}
\tag{1.107}
$$

Under the supersymmetry transformations the action changes by by the total derivative

$$\delta S = \int dt \left(i\epsilon\dot{\mathcal{Q}} + i\bar{\epsilon}\dot{\bar{\mathcal{Q}}} \right) \tag{1.108}$$

which can be used to derive the phase space generators of supersymmetry

$$\begin{aligned} \mathcal{Q} &= ip_j\bar{\psi}^j \\ \bar{\mathcal{Q}} &= -ip_j\psi \end{aligned} \tag{1.109}$$

The SUSY transformation for arbitrary function $F(p, x, \psi, \bar{\psi})$ on phase space can be expressed via

$$\delta F = \{F, i\epsilon\mathcal{Q} + i\bar{\epsilon}\bar{\mathcal{Q}}\}_{pb}. \tag{1.110}$$

The nonzero Poisson brackets of coordinate functions for our system are of the form

$$\{p_k, x^j\}_{pb} = \delta^j_k, \quad \{\psi^j, \bar{\psi}^k\}_{pb} = i\delta^{jk}. \tag{1.111}$$

In canonical quantization Poisson brackets are promoted to commutators

$$[\hat{p}_k, \hat{x}^j] = -i\delta^j_k, \quad \{\hat{\psi}^j, \hat{\bar{\psi}}^k\} = \delta^{jk}. \tag{1.112}$$

Let us chose vertical polarization for even part of the phase space and holomorphic polarization for the odd one. The basis $|\psi, x\rangle$ in Hilbert space in this polarization are given by the eigenstates of $\hat{\psi}$ and $\hat{x}$

$$\hat{\psi}|\psi, x\rangle = \psi|x, \psi\rangle, \qquad \hat{x}|x, \psi\rangle = x|x, \psi\rangle. \tag{1.113}$$

This basis is complete

$$1_{\mathcal{H}} = \int d^n x d^{2n}\psi \;\; e^{\sum \psi^j\bar{\psi}^j} \;\; |x, \psi\rangle\langle x, \bar{\psi}|, \tag{1.114}$$

so we can identify

$$\mathcal{H} = C^\infty(\mathbb{R}^{n|n}). \tag{1.115}$$

The inner product in terms of wavefunctions

$$\begin{aligned} \langle\Psi|\Phi\rangle &= \int d^n x d^{2n}\psi \;\; e^{\sum \psi^j\bar{\psi}^j} \langle\Psi|x, \psi\rangle\langle x, \bar{\psi}|\Phi\rangle \\ &= \int d^n x d^n\psi d^n\bar{\psi} \;\; e^{\sum \psi^j\bar{\psi}^j} \;\; \Psi(x, \psi)\overline{\Phi(x, \psi)} = \langle\Psi, \Phi\rangle \end{aligned} \tag{1.116}$$

In previous sections we discussed that we can identify differential forms on $\mathbb{R}^n$ and functions on supermanifold $\mathbb{R}^{n|n}$

$$\Omega^*(\mathbb{R}^n) \to C^\infty(\mathbb{R}^{n|n}) : \omega \mapsto F_\omega \tag{1.117}$$

The inner product on functions

$$\begin{aligned}\langle F_\omega, F_\mu \rangle &= \int d^n x d^n \psi d^n \bar{\psi} \;\; e^{\sum \psi^j \bar{\psi}^j} \;\; F_\omega(x,\psi)\overline{F_\mu(x,\psi)} \\ &= (-1)^{(\ldots)} \int d^n x d^n \psi \;\; F_\omega(x,\psi) F_{*\mu}(x,\psi) = (-1)^{(\ldots)} \int_{\mathbb{R}^{n|n}} d^n x d^n \psi \;\; F_{\omega \wedge *\mu}(x,\psi) \\ &= (-1)^{(\ldots)} \int_{\mathbb{R}^n} \omega \wedge *\mu = (-1)^{(\ldots)} \langle \mu, \omega \rangle\end{aligned} \tag{1.118}$$

matches with the Hodge product for differential forms. The supercharges $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ become operators in quantum theory. Using the $|\psi, x\rangle$ basis we can express them as differential operators acting on wave-functions

$$\begin{aligned}\mathcal{Q} = i \sum p_j \bar{\psi}^j \mapsto \hat{\mathcal{Q}} = Q = \sum \frac{\partial}{\partial \psi^j} \frac{\partial}{\partial x^j} \\ \bar{\mathcal{Q}} = -i \sum p_j \psi^j \mapsto \hat{\bar{\mathcal{Q}}} = \bar{Q} = \sum \psi^j \frac{\partial}{\partial x^j}\end{aligned} \tag{1.119}$$

The operators Q and $\bar{Q}$ are external derivative d and its Hodge dual d^*. Indeed if we use wavefunction F_ω associated with the differential form ω

$$\bar{Q} F_\omega = F_{d\omega}, \quad Q F_\omega = (-1)^{(\ldots)} F_{d^*\omega}. \tag{1.120}$$

The Hamiltonian is the Hodge-Laplacian operator

$$\hat{H} = \{Q, \bar{Q}\} = \{d, d^*\} = \Delta. \tag{1.121}$$

The supersymmetric ground states

$$\hat{H}|\omega\rangle = |\Delta\omega\rangle = 0 \tag{1.122}$$

are harmonic forms and are in one-to-one correspondence with de Rham cohomology! In next sections we will use various SQM methods to describe approximate ground states by deforming the theory with superpotential. The simplest way of describing the supersymmetric deformations is to use the superspace formalism.

1.7 $d = 1$ $N = 2$ Superspace formalism

Let us consider 3d superspace $\mathbb{R}^{1|2}$ with even coordinate t and odd coordinates θ and $\bar{\theta}$. The maps

$$\mathbb{R}^{1|2} \to \mathbb{R} \ : \ (t, \theta, \bar{\theta}) \mapsto x(t, \theta, \bar{\theta}) \tag{1.123}$$

are functions $x(t, \theta, \bar{\theta})$ on superspace, which are finite polynomials in odd variables

$$Map(\mathbb{R}^{1|2}, \mathbb{R}) = \mathbb{R}^{2|2} \otimes C^{\infty}(\mathbb{R}) \tag{1.124}$$

We can use real functions $x(t), F(t)$ and complex functions $\psi(t), \bar{\psi}(t)$ to expand the superfield

$$x(t, \theta, \bar{\theta}) = x(t) + \theta\bar{\psi}(t) - \bar{\theta}\psi(t) + F(t)\theta\bar{\theta}. \tag{1.125}$$

We choose our expansion coefficients, so that $x(t, \theta, \bar{\theta})$ is real superfield i.e.

$$\begin{aligned} x^{\dagger}(t, \theta, \bar{\theta}) &= x^{\dagger} + (\theta\bar{\psi})^{\dagger} - (\bar{\theta}\psi)^{\dagger} + (F\theta\bar{\theta})^{\dagger} \\ &= x + \psi\bar{\theta} - \bar{\psi}\theta + F\theta\bar{\theta} = x - \bar{\theta}\psi + \theta\bar{\psi} + F\theta\bar{\theta} \end{aligned}$$

Remark: We can write the real superfiel x in more symmetric form

$$\hat{x}(t, \theta, \bar{\theta}) = x(t) + \theta\bar{\psi}(t) + \psi(t)\bar{\theta} + F(t)\theta\bar{\theta}$$

Similarly to $d = 0$ $N = 2$ we want to choose diffeomorphisms of $\mathbb{R}^{1|2}$ which obey the $N = 2$ $d = 1$ algebra. In addition to the familiar translations in θ and $\bar{\theta}$ we add rotations in t so that the infinitesimal transformations take the form

$$\begin{aligned} &t \to t + c + i\bar{\epsilon}\theta + i\epsilon\bar{\theta} \\ &\theta \to \theta + \epsilon, \quad \bar{\theta} \to \bar{\theta} + \bar{\epsilon} \end{aligned} \tag{1.126}$$

The infinitesimal transformations are generated by vector fields

$$\begin{aligned} \mathfrak{Q} &= \frac{\partial}{\partial\theta} + i\bar{\theta}\frac{\partial}{\partial t}, \\ \bar{\mathfrak{Q}} &= \frac{\partial}{\partial\bar{\theta}} + i\theta\frac{\partial}{\partial t} \\ \mathfrak{H} &= \frac{\partial}{\partial t} \end{aligned} \tag{1.127}$$

i.e for arbitrary function $F(t,\theta,\bar\theta)$

$$\delta F(t,\theta,\bar\theta) = (\epsilon\mathfrak{Q} + \bar\epsilon\bar{\mathfrak{Q}})F(t,\theta,\bar\theta) \tag{1.128}$$

Generators $\mathfrak{Q}, \bar{\mathfrak{Q}}, \mathfrak{H}$ form the $d = 1$ $N = 2$ algebra

$$\begin{aligned} \{\mathfrak{Q},\mathfrak{Q}\} &= \{\bar{\mathfrak{Q}},\bar{\mathfrak{Q}}\} = 0 \\ \{\mathfrak{Q},\bar{\mathfrak{Q}}\} &= -2i\mathfrak{H} \end{aligned} \tag{1.129}$$

The (pullback) action of the SUSY algebra on the the space of maps

$$\begin{aligned} \delta_\epsilon x(\theta,\bar\theta) &= x(t + i\bar\epsilon\theta + i\epsilon\bar\theta, \theta+\epsilon, \bar\theta+\bar\epsilon) - x(t,\theta,\bar\theta) \\ &= (\epsilon\mathfrak{Q} + \bar\epsilon\bar{\mathfrak{Q}})x(t,\theta,\bar\theta) = i(\bar\epsilon\theta + \epsilon\bar\theta)\dot x + \epsilon\bar\psi - \bar\epsilon\psi + i\theta\epsilon\bar\theta\dot{\bar\psi} - i\bar\theta\bar\epsilon\theta\dot\psi + \theta\bar\epsilon F + \epsilon\bar\theta F \\ &= (\epsilon Q + \bar\epsilon Q)x(t,\theta,\bar\theta) = \delta_\epsilon x + \theta\delta_\epsilon\bar\psi - \bar\theta\delta_\epsilon\psi + \theta\bar\theta\delta_\epsilon F \end{aligned} \tag{1.130}$$

The action in components

$$\begin{aligned} \delta_\epsilon x &= (\epsilon Q + \bar\epsilon\bar Q)x = \epsilon\bar\psi - \bar\epsilon\psi \\ \delta_\epsilon \psi &= (\epsilon Q + \bar\epsilon\bar Q)\psi = \epsilon(i\dot x + F) \\ \delta_\epsilon \bar\psi &= (\epsilon Q + \bar\epsilon\bar Q)\bar\psi = \bar\epsilon(-i\dot x + F) \\ \delta_\epsilon F &= (\epsilon Q + \bar\epsilon\bar Q)F = -i\epsilon\dot{\bar\psi} - i\bar\epsilon\dot\psi \end{aligned} \tag{1.131}$$

Remark: The SUSY transformations for t-independent functions is identical to the $d = 0$ $N = 2$ transformations we discussed before.

Similarly to $d = 0$ case we can introduce supercovarint derivatives

$$\begin{aligned} \mathfrak{D} &= \frac{\partial}{\partial\theta} - i\bar\theta\frac{\partial}{\partial t}, \\ \bar{\mathfrak{D}} &= \frac{\partial}{\partial\bar\theta} - i\theta\frac{\partial}{\partial t} \\ \{\mathfrak{Q},\mathfrak{D}\} = \{\bar{\mathfrak{Q}},\mathfrak{D}\} &= \{\mathfrak{Q},\bar{\mathfrak{D}}\} = \{\bar{\mathfrak{Q}},\bar{\mathfrak{D}}\} = 0 \end{aligned} \tag{1.132}$$

to create other type of superfields. In particular and integral over superspace $\mathbb{R}^{1|2}$ of the arbitrary functions of superfields and covariant derivatives

$$S[x, F, \psi, \bar{\psi}] = \int dtd\theta d\bar{\theta} \ \ \mathcal{F}(x(t,\theta,\bar{\theta}), \mathfrak{D}x(t,\theta,\bar{\theta}), \bar{\mathfrak{D}}x(t,\theta,\bar{\theta})) \tag{1.133}$$

is invariant under the action of supersymmetry transfromations (1.131). The SUSY variation of the Lagrangian

$$\begin{aligned}
\delta_\epsilon L = (\epsilon Q + \bar{\epsilon}\bar{Q})L &= (\epsilon Q + \bar{\epsilon}\bar{Q}) \int d\theta d\bar{\theta} \ \ \mathcal{F} \\
&= \int d\theta d\bar{\theta} \ \ (\epsilon Q + \bar{\epsilon}\bar{Q})\mathcal{F} = \int d\theta d\bar{\theta} \ \ (\epsilon\mathfrak{Q} + \bar{\epsilon}\bar{\mathfrak{Q}})\mathcal{F} \\
&= \int d\theta d\bar{\theta} \ \ (\epsilon(\partial_\theta + i\bar{\theta}\partial_t) + \bar{\epsilon}(\partial_{\bar{\theta}} + i\theta\partial_t))\mathcal{F}(x(\theta,\bar{\theta}), \mathfrak{D}x(\theta,\bar{\theta}), \bar{\mathfrak{D}}x(\theta,\bar{\theta})) \\
&= i\partial_t \int d\theta d\bar{\theta} \ \ (\epsilon\bar{\theta} + \bar{\epsilon}\theta)\mathcal{F}(x(\theta,\bar{\theta}), \partial_\theta x, \partial_{\bar{\theta}}\hat{x}) = \partial_t K(x, \psi, F).
\end{aligned} \tag{1.134}$$

We used the Berezin integration rules to eliminate Grassmann integrals of total derivatives.

1.8 Superspace description of harmonic oscillator

The superspace Lagrangian for supersymmetric harmonic oscillator

$$L = \frac{1}{2}\int d\theta d\bar{\theta} \ \ \left(\mathfrak{D}x\bar{\mathfrak{D}}x - \omega x^2\right) \tag{1.135}$$

Explicit superspace integration for kinetic

$$\begin{aligned}
&\frac{1}{2}\int d\theta d\bar{\theta} \ \ \mathfrak{D}x\bar{\mathfrak{D}}x = \frac{1}{2}\int d\theta d\bar{\theta} \ \ (\partial_\theta - i\bar{\theta}\partial_t)x(\partial_{\bar{\theta}} - i\theta\partial_t)x \\
&= \frac{1}{2}\int d\theta d\bar{\theta} \ \ (\bar{\psi} + \bar{\theta}F - i\bar{\theta}\dot{x} - i\bar{\theta}\theta\dot{\bar{\psi}})(-\psi - \theta F - i\theta\dot{x} + i\theta\bar{\theta}\dot{\psi}) \\
&= \frac{1}{2}\int d\theta d\bar{\theta} \ \ (i\bar{\psi}\theta\bar{\theta}\dot{\psi} - \bar{\theta}\theta F^2 - \bar{\theta}\theta\dot{x}^2 + i\bar{\theta}\theta\dot{\bar{\psi}}\psi) \\
&= \frac{1}{2}(i\bar{\psi}\dot{\psi} + F^2 + \dot{x}^2 - i\dot{\bar{\psi}}\psi)
\end{aligned} \tag{1.136}$$

and potential

$$
\begin{aligned}
-\frac{1}{2}\int d^2\theta\, \omega x^2 &= -\frac{\omega}{2}\int d^2\theta\ (x+\theta\bar{\psi}-\bar{\theta}\psi+F\theta\bar{\theta})^2 \\
&= -\omega\int d^2\theta\ \big(xF\theta\bar{\theta}-\theta\bar{\psi}\bar{\theta}\psi\big) \\
&= -\omega F x-\omega\bar{\psi}\psi
\end{aligned}
\tag{1.137}
$$

terms allows us to express Lagrangian

$$L=\frac{1}{2}(i\bar{\psi}\dot{\psi}+F^2+\dot{x}^2-i\dot{\bar{\psi}}\psi)-\omega F x-\omega\bar{\psi}\psi \tag{1.138}$$

in terms of components of the superfield

$$x(t,\theta,\bar{\theta})=x(t)+\theta\bar{\psi}(t)-\bar{\theta}\psi(t)+F(t)\theta\bar{\theta}. \tag{1.139}$$

Integration out F in path integral is the same as solving equations of motion for F in terms of the other fields

$$\frac{\delta L}{\delta F}=F-\omega x=0, \tag{1.140}$$

so the Lagrangian

$$L=\frac{1}{2}(i\bar{\psi}\dot{\psi}-\omega^2x^2+\dot{x}^2-i\dot{\bar{\psi}}\psi)-\omega\bar{\psi}\psi \tag{1.141}$$

becomes the harmonic oscillator Lagrangian from previous section.