Supersymmetric Quantum Mechanics and Morse theory

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Lectures 1,2

Course introduction. Topological invariants. Simplicial models

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1 Course overview

During term 3 of 2021 year I will give a special topic course " Supersymmetric Quantum Mechanics and Morse theory" to graduate students at OIST. The goal of the course is to give an introduction to the modern mathematical physics, the research area on the intersection of modern physics and math. It is rapidly developing now as it can use the ideas and methods from both physics and math. In order to do research in this area one typically needs a dual math-physics education, which is quite rare. In my course I want to illustrate the advantages of dual description using Morse theory as an example to hopefully inspire some students to take a difficult path of dual math-physics education.

It so happens that in parallel to my course the graduate school at OIST offers a special course "An Introduction to Supersymmetric Field Theories and (Super)Strings" so we will spend some extra time on topics that might be useful for this course: the superspace description of the supersymmetric quantum mechanics and the topological classification of the 2d manifolds via Morse theory.

1.1 Course information

Title: Supersymmetric Quantum Mechanics and Morse theory

Instructor: Vyacheslav Lysov (Neiman Unit)

Duration: 13 weeks, 1 hour lectures, twice a week.

Meeting time: Tuesday, Thursday, afternoon, 2-3 pm.

Description: Modern theoretical physics has deep connection with modern mathematics. The foundations of this connection was established by Edward Witten in his works around 80's. We will use Witten‘s paper "Supersymmetry and Morse theory" as prime reference for the course. It is a scientific paper, so it assumes some background knowledge of both supersymmetry and Morse theory, which we will cover in the first part of the course. The second part of the course will be focused on two different approach to the Morse theory: the standard mathematical one, based on differential geometry and topology and the physical one using the supersymmetric quantum mechanics. The goal of this course is to demonstrate the close relation between the modern physics in mathematics using the simplest possible example.

Background: Some knowledge of quantum mechanics and differential geometry.

Grading: There will be biweekly home assignments.

Office hours: You can find me in my office (L4E25e) most of the days before noon. I sometimes have seminars Zoom meetings after 2-3 pm so morning is the time when I free to talk everyday.

Logistics: There is a course website as a part of Quantum Gravity Unit website. I will update the lecture notes twice a week. The homework will be posted on course website.

References:

- E. Witten, "Supersymmetry and Morse theory," J. Diff. Geom. **17** (1982) no.4, 661-692
- Hori, K., Thomas, R., Katz, S., Vafa, C., Pandharipande, R., Klemm, A., Vakil, R. and Zaslow, E., 2003. Mirror symmetry (Vol. 1). American Mathematical Soc..
- A. Hatcher, "Algebraic topology"

- R. Bott, L. W. Tu, “Differential Forms in Algebraic Topology”

1.2 Lecture plan

1. Introduction to topological invariants and cohomology.
2. Differential forms and de Rham cohomology.
3. Quantum mechanics review.
4. Grassmann variables.
5. Supersymmetric quantum mechanics.
6. Superspace formalism.
7. Supersymmetric sigma model as a de Rham theory.
8. Morse theory.
9. Morse complex for de Rham cohomology.
10. Tunneling in quantum mechanics.
11. Instantons in supersymmetric sigma model.
12. Generalizations of Morse theory

2 Topological invariants

There are two notions of topology:

- **Shape**: *Topology* is a collection of the properties of a geometric object that are invariant under the continuous deformations, such as stretching, crumping, bending, twisting.e.t.c. The examples of non-continuous deformations are gluing, tearing, open holes and closing holes.
- **Closeness**: A *topological space* is a pair (X, τ), where X is a set and τ is a collection of subsets of X such that
    1. The empty set $\emptyset$ and X itself belong to τ.
    2. Any arbitrary (finite or infinite) union of members of τ still belongs to τ.

3. The intersection of any finite number of members of τ still belongs to τ.

The elements of τ are called *open sets* and the collection τ is called a *topology* on X.

Example: For finite set $X = \{1, 2, 3\}$ we can use several topologies

- The topology $\tau = \{\emptyset, \{1, 2, 3\}\} = \{\emptyset, X\}$ is know as the *trivial topology*.
- The topology $\tau = \{\emptyset, \{2\}, \{2, 3\}, \{1, 2, 3\}\}$.

Example: Let K be an algebraically closed field then we can define n-dimensional affine space $\mathbb{A}^n$ as a set formed by n-tuples of elements in K. *Zariski topology* on $\mathbb{A}^n$ describes open sets as complements of closed sets of the form

$$V(S) = \{x \in \mathbb{A}^n \mid f(x) = 0, \forall f \in S\} \tag{2.1}$$

where S is the set of polynomials over n variables with coefficients in K.

Example: We can describe 1d topological spaces by specifying the topology near the boundary points. We can embed some number of open intervals into $\mathbb{R}$ and use the $\mathbb{R}$-topology for internal points, while explicitly specifying topology at the boundary. For a single open interval $(0, 1)$ there are two possibilities

- **Interval**: The boundary points $\{0, 1\}$ are special points, i.e true boundary. The topological space is the space with the boundary - the interval $I = [0, 1]$.
- **Circle**: The boundary points are in neighborhood of each other so we can glue the open interval into a topological space with no boundary - a circle S^1 also known as the 1d sphere.

Example: In case of two open intervals $(0, 1)$ and $(2, 3)$ there are 5 possible topologies (up to relabeling points):

- **Two intervals**: All boundary points $\{0, 1, 2, 3\}$ are true boundary. The topological space is the disjoint union of two intervals $I_1 \sqcup I_2$.
- **Interval**: The boundary points $\{0, 3\}$ are true boundary while $\{1\}$ and $\{2\}$ are identified. The topological space is the interval $I = [0, 3]$.
- **Circle**: We identify two pair of boundary points: $\{1\}$ and $\{2\}$, additionally $\{0\}$ and $\{3\}$. The topological space is the circle S^1.

- **To circles**: We identify two pair of boundary points: $\{0\}$ and $\{1\}$, additionally $\{2\}$ and $\{3\}$. The topological space is disjoint union of two circles $S^1 \sqcup S^1$.
- **Circle and interval**: The boundary points $\{0, 1\}$ are true boundary while $\{2\}$ and $\{3\}$ are identified. The topological space is disjoint union of circle and interval $I \sqcup S^1$.

Example: Let us consider open square $(0, 1) \times (0, 1)$ embedded into $\mathbb{R}^2$ so we need to specify only the topology for boundary points. There several possible topologies:

- **Disc**: All boundary points are true boundary. The topological space is the 2d space with the boundary - the disc D^2, also known as 2d ball B^2
- **Sphere**: All boundary points are in neighborhood of each other so we can glue all of them into a topological space with no boundary - 2d sphere S^2.
- **Cylinder**: Let us decompose boundary points on the 4 components: two vertical and two horizontal. Let us identify two horizontal components of the boundary while preserving the 1d topology along them. The resulting topological space is the cylinder $S^1 \times I$.
- **Torus**: Let us start with the cylinder and further identify its two boundaries while preserving the 1d topology on S^1. The resulting topological space is torus $T^2 = S^1 \times S^1$.

2.1 Topological invariants

We can describe topology using *topological invariants* - some functions of geometry invariant under continuous deformations. In our course we will focus on the most basic topological invariants:

- **number of connected components**: The simplest and the most intuitive invariant. It is preserved as long as we do not do cutting and gluing.
- **Euler characteristic**: Quite popular in physics as various perturbative series involve summation over graphs, ribbon graphs or Riemann surfaces. Graphs of the same topology (number of loops) describe particular type of physical quantity: tree graphs - classical contribution, 1-loop graphs - leading $\hbar$-correction e.t.c. Similarly in string theory the contribution of the string coupling is weighted by the Euler characteristics of the diagram.

- **de Rham cohomology**: A generalization and refinement of the number of connected components and Euler characteristics to higher dimensions. widely used in physics and math.

Example: 0d topological spaces are finite collection of points, which can be classified by a single integer the number of points, which matches with zeroth Betti number

$$b_0\left(\bigsqcup_{i=1}^{n}\bullet\right) = n \tag{2.2}$$

Example: 1d topological spaces (manifolds actually) are finite disjoint union of intervals and circles, so we can classify them by two integers, zeroth Betti number and Euler characteristic

$$M_{m,n} = \bigsqcup_{i=1}^{n} S^1 \sqcup \bigsqcup_{j=1}^{m} I, \quad b_0(M_{m,n}) = m+n, \quad \chi(M_{m,n}) = m \tag{2.3}$$

Example: The 2d connected topological spaces without boundary can be classified in terms of single integer - genus, equivalently in terms of Euler characteristics

$$\chi(\Sigma_g) = 2 - 2g. \tag{2.4}$$

Example: Let us provide values for topological invariants for our examples

X	b_0	χ	b_1	b_2
pt	1	1	0	0
I	1	1	0	0
S^1	1	0	1	0
$I \sqcup I$	2	2	0	0
$S^1 \sqcup I$	2	1	1	0
$S^1 \sqcup S^1$	2	0	2	0
D^2	1	1	0	0
S^2	1	2	0	1
$S^1 \times I$	1	0	1	0
T^2	1	0	2	1

(2.5)

Problem: The geometric objects of our interest are smooth manifolds. The smoothness is great when we consider a single object such as function on smooth manifold. However, the Euler characteristic and cohomology in smooth case are defined as graded dimension and quotient for the space of smooth objects on manifold. Due to smoothens of the individual elements of such spaces they have infinite dimension, hence require very careful analysis.

Resolution: The physics approach to this problem would be to find a finite-dimensional model for cohomology such the the limit of very large dimension matches with continuous description. We can construct a finite-dimensional model of smooth manifold M by cutting it into a universal pieces. The universal pieces have the triangular shape in 2d so such approach in physical literature is often called the *triangulation* of M. From more careful mathematical perspective the triangulation of M is the image of the map $\sigma : \Delta_M \to M$ for a *simplicial model* Δ_M of M. Algebraic topology tells us that the cohomology of the triangulation, also known as the *singular cohomology* are identical to the ones for simplicial model. In our brief introduction to cohomology we will focus on the simplicial homology and cohomology, while rely on the algebraic topology results to justify that they match with the singular cohomology and have good continuous limit.

2.2 Euler characteristic for triangulation

The simplest triangulation Δ_{S^1} for S^1 consists of three edges and three vertices

$$V = E = 3, \tag{2.6}$$

while a generic triangulation of S^1 may have arbitrary number of edges, but the number of vertices always equal to it

$$V = E \tag{2.7}$$

The Euler characteristic of a 1d triangulation is

$$\chi(\Delta) = V - E \tag{2.8}$$

In case of a simplest triangulation

$$\chi(\Delta_{S^1}) = V - E = 3 - 3 = 0 \tag{2.9}$$

Moreover due to the fact the the number of vertices and number of edges match for arbitrary triangulation of a circle the Euler characteristics is independent on the choice of triangulation, so it describes a topology of S^1, rather then features of particular model for S^1. We can write

$$\chi(S^1) = \chi(\Delta_{S^1}) = 0 \tag{2.10}$$

Furthermore we can algorithmically go from one triangulation to another using refining and coarsing procedures. Both of them a local i.e involve some operations on a simplex and its neighbors.

- The refining procedure is a split of the edge in two by adding a vertex in the middle

$$\Delta_{S^1} = \bullet - \bullet \quad \mapsto \quad \Delta'_{S^1} = \bullet - \bullet - \bullet \tag{2.11}$$

  The change in the edge and vertex numbers

$$\delta V = V' - V = +1, \;\; \delta E = E' - E = +1, \quad \delta\chi = \delta V - \delta E = 0. \tag{2.12}$$

- The coarsing operation is a removal of vertex and connecting two attached edges onto a single one

$$\bullet - \bullet - \bullet \quad \mapsto \quad \bullet - \bullet. \tag{2.13}$$

  The change in the edge and vertex numbers

$$\delta V = -1, \;\; \delta E = -1, \quad \delta\chi = \delta V - \delta E = 0 \tag{2.14}$$

The 2d case is probably familiar for everyone

$$\chi(\Delta) = V - E + F \tag{2.15}$$

Conclusion: Euler characteristic is a topological invariant of the manifold, so it can be computed using a "simple model" of a manifold in a form of triangulation. The independence on the triangulation allows us to consider a *continuous limit* of the Euler characteristic where triangulations become a smooth structure on a manifold.

3 Cohomology theory

Homology is another example of the topological invariant of the manifold that has a straightforward descriptions in terms of *simplicial model* of the manifold constructed via *triangulation.*

3.1 Simplicial complex

Definition: n-simplex is an n-dimensional polytope which is convex hull of $n+1$ vertices. More explicitly, let collection of points $u_0, ...u_k \in \mathbb{R}^k$ be such that $u_1 - u_0, ..., u_k - u_0$ are linearly independent then they define the k-simplex

$$\Delta^k = \{t_0 u_0 + ... + t_k u_k | \sum t_i = 1, \;\; t_i \geq 0, \; i = 0, ..., k\} \tag{3.1}$$

We will use the $e_{0...k}$ notation for the simplex with vertices $0, .., k$. Simplex $e_{0...k}$ naturally contains smaller simplexes, obtained by removing some of the points. The points of $e_{0...k}$ with $t_k = 0$ form a simplex of dimension $k-1$ for points $u_0, .., u_{k-1}$ which we can label as $e_{0...k-1}$. The $e_{0...k-1}$ is obtained from $e_{0...k}$ by the removal of the point u_k. We can remove more than one point hence constructing the the lower dimensional simplices.

Definition: On a single simplex we can define a boundary operation

$$\partial e_{a_1 a_2 ... a_k} = e_{a_2 a_3 ... a_k} - e_{a_1 a_3 ... a_k} + ... + (-1)^{k+1} e_{a_1 ... a_{k-1}} \tag{3.2}$$

which is nothing but a geometrical boundary of simplex $e_{a_1...a_k}$ defined in terms of sub-simplices. The signs are required to properly describe the relative orientation.

Example: The 1-simplex is a line segment e_{01} with two 0-sub-simplices e_0, e_1 that are the end points, while the boundary

$$\partial e_{01} = e_1 - e_0, \;\; \partial e_0 = \partial e_1 = 0 \tag{3.3}$$

Example: The 2-simplex is a triangle e_{012} with three 1-sub-simplices e_{01}, e_{12}, e_{02} and three 0-simplices e_0, e_1, e_2

$$\partial e_{012} = e_{12} - e_{02} + e_{01}, \;\; \partial e_{ij} = e_j - e_i, \;\; \partial e_i = 0 \tag{3.4}$$

Let us notice that for our example

$$\partial^2 e_{012} = \partial(e_{12} - e_{02} + e_{01}) = e_2 - e_1 - (e_2 - e_0) + e_1 - e_0 = 0, \tag{3.5}$$

moreover

$$\partial^2 e_{ij} = \partial(e_j - e_i) = 0, \quad \partial^2 e_i = \partial(0) = 0 \tag{3.6}$$

so $\partial^2 = 0$ on all subsimilices of a 2-simplex.

Proposition: The boundary operator squares to zero i.e. "boundary of boundary is trivial"

$$\partial \circ \partial = 0 \tag{3.7}$$

Proof: Evaluation on k-simplex

$$\begin{aligned}
\partial^2 e_{a_1 a_2 \dots a_k} &= \partial \sum_{i=1}^{k} (-1)^{i+1} e_{a_1 a_2 .. \hat{a}_i \dots a_k} \\
&= \sum_{i=1}^{k} (-1)^{i+1} \sum_{j=1}^{i} (-1)^{j+1} e_{a_1 a_2 .. \hat{a}_j .. \hat{a}_i \dots a_k} + \sum_{i=1}^{k} (-1)^{i+1} \sum_{j=i+1}^{k} (-1)^{j} e_{a_1 a_2 .. \hat{a}_i \dots \hat{a}_j \dots a_k} \\
&= \sum_{j<i, i=1}^{k} (-1)^{i+j} e_{a_1 a_2 .. \hat{a}_j .. \hat{a}_i \dots a_k} + \sum_{i<j, j=1}^{k} (-1)^{i+j+1} e_{a_1 a_2 .. \hat{a}_i \dots \hat{a}_j \dots a_k} \\
&= \sum_{j<i, i=1}^{k} (-1)^{i+j} e_{a_1 a_2 .. \hat{a}_j .. \hat{a}_i \dots a_k} - \sum_{i<j, j=1}^{k} (-1)^{i+j} e_{a_1 a_2 .. \hat{a}_i \dots \hat{a}_j \dots a_k} = 0.
\end{aligned} \tag{3.8}$$

Construction: The *simplicial model* Δ_M of n-dimensional manifold M is a collection of n-simplexes Δ^n_α which we glue along the codimension-1 subsimplices

$$\Delta_M = \bigcup_{\Delta^{n-1}} \Delta^n_\alpha. \tag{3.9}$$

We can define linear spaces $C_k(\Delta_M)$ as a linear span of all k-simplexes of the simplicial model Δ_M i.e

$$C_k(\Delta_M) = \mathbb{R}\langle e_{i_1 \dots i_k} \rangle, \quad e_{i_1 \dots i_k} \in \Delta_M. \tag{3.10}$$

The boundary operation is linear i.e.

$$\partial(a e_{ij} + b e_{kl}) = a\, \partial e_{ij} + b\, \partial e_{kl}, \tag{3.11}$$

so we can extend to spaces C_k to define a series of linear maps

$$\partial : \quad C_k(\Delta_M) \to C_{k-1}(\Delta_M). \tag{3.12}$$

Observation: A finite-dimensional model of a smooth manifold M, the simplicial model Δ_M allows us to define a sequence of vector spaces and linear maps between them which form a *simplicial complex* $(C_k(M), \partial)$.

Example: The simplest manifold is a point $M = pt$ with a triangulation being a single 0-simplex

$$\Delta_{pt} = \Delta^0 = \bullet_0. \tag{3.13}$$

There is a single vector space for 0d simplex

$$C_0 = \mathbb{R}\langle e_0 \rangle = \mathbb{R}. \tag{3.14}$$

The chain complex is

$$C_\bullet(\Delta_0) = \qquad \mathbb{R} \xrightarrow{\partial_0} 0. \tag{3.15}$$

Example: The simplicial model for an interval $I = [0, 1]$ is a single 1d simplex

$$\Delta_I = \Delta^1 = \bullet_0 - \bullet_1. \tag{3.16}$$

The 1d simplex Δ^1, labeled as e_{01} has two 0d subsimplices e_0 and e_1, so the corresponding vector spaces

$$C_0 = \mathbb{R}\langle e_0, e_1 \rangle = \mathbb{R}^2, \;\; C_1 = \mathbb{R}\langle e_{01} \rangle = \mathbb{R}. \tag{3.17}$$

The boundary operation acts on C_1 in the following way

$$\partial(c_{01}e_{01}) = c_{01}\partial e_{01} = c_{01}(e_1 - e_0) = -c_{01}e_0 + c_{01}e_1. \tag{3.18}$$

The chain complex for Δ_I is of the form

$$C_\bullet = \qquad \mathbb{R} \xrightarrow{\partial_1} \mathbb{R}^2 \xrightarrow{\partial_0} 0 \tag{3.19}$$

$$c_{01} \longrightarrow (-c_{01}, c_{01})$$