Melonic large N limit of 5-index irreducible random tensors

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NORDITA
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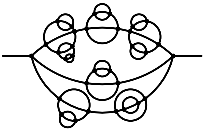
History of random tensors

- First introduced in zero dimension: random geometry and quantum gravity [Ambjorn Durhuus Jonsson '90, Boulatov '92, Ooguri '92, ...]
- Strongly coupled QFTs and holography ($d = 1$): SYK model without disorder [Witten, Klebanov, Tarnopolsky, ...]
- Tensor models in higher dimension: new class of conformal field theories $\rightarrow$ melonic CFTs



Different types of melonic limit

- Multiple fields: 4 tensor fields, $O(N)^6$ symmetry

$$T^i_{abc}$$

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- SYK model

$$J_{i_1 i_2 i_3 i_4} \psi_{i_1} \psi_{i_2} \psi_{i_3} \psi_{i_4}$$

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- Multi matrix models with a large number of matrices [Ferrari, Schaposnik Massolo, Valette ...], different symmetry groups

Colored $O(N)$ models

$T_{a_1 a_2 \dots a_r}$ in fundamental representation of $O(N) \times O(N) \times \cdots \times O(N)$

- Propagator: $P_{a_1 a_2 \dots a_r, b_1 b_2 \dots b_r} = \delta_{a_1 b_1} \delta_{a_2 b_2} \dots \delta_{a_r b_r}$
- Interaction: complete graph K_{r+1}

K_4 K_6

Theorem

A melonic large N limit exists for prime r. Ferrarri, Rivasseau, Valette '17

Irreducible tensor models

- What about other tensor representations?
- Completely symmetric tensors: no melonic large N limit
- Irreducible tensors:
  - Propagator: orthogonal projector on an irreducible representation of $O(N)$
  - Interaction: Complete graph invariant

- Do these models admit a large N expansion?
- Is it melonic?

Irreducible tensor models

Conjecture:
For $r = 3$, there exists a melonic large N limit for $O(N)$ symmetric traceless tensors. [Klebanov, Tarnopolsky '17]

- Evidence: Explicit numerical check for all diagrams up to order λ^8
- Proof and generalizations:
  - $O(N)$ irreducible, $r = 3$ [Benedetti, Carrozza, Gurau, Kolanowski]
  - $Sp(N)$ irreducible, $r = 3$ [Carrozza, Pozsgay]
- Generalization for $r > 3$?

$\Rightarrow$ Here for 5 indices

Outline

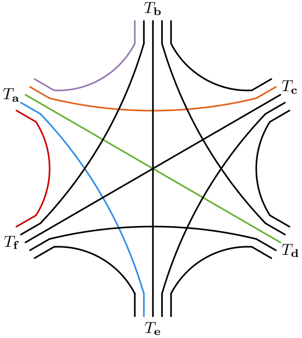
The model

Free energy:

$$F_{\boldsymbol{P}}(\lambda) = \frac{6}{N^5}\lambda\partial_\lambda \ln\left\{ \left[e^{\frac{1}{2}\partial_T \boldsymbol{P} \partial_T} e^{\frac{\lambda}{6N^5}\delta^h_{\boldsymbol{abcdef}} T_{\boldsymbol{a}} T_{\boldsymbol{b}} T_{\boldsymbol{c}} T_{\boldsymbol{d}} T_{\boldsymbol{e}} T_{\boldsymbol{f}}} \right]_{T=0} \right\}$$

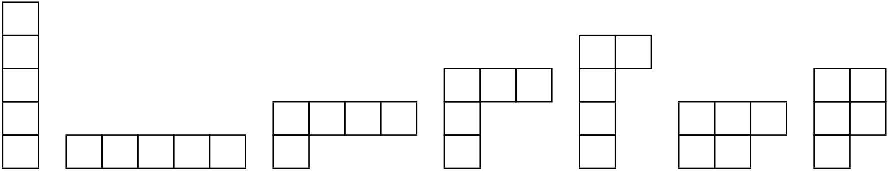
- $\boldsymbol{P}$ one of seven orthogonal projectors on irreducible representations
- Interaction vertex

$$T_{a_1 a_2 a_3 a_4 a_5} T_{a_5 b_2 b_3 b_4 b_5} T_{b_5 a_4 c_3 c_4 c_5} T_{c_5 b_4 a_3 d_4 d_5} T_{d_5 c_4 b_3 a_2 e_5} T_{e_5 d_4 c_3 b_2 a_1}$$



Propagator

$\boldsymbol{P}$: orthogonal projector on one of the irreducible tensor spaces



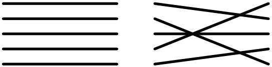
For traceless symmetric tensors:

$$\boldsymbol{S}_{\boldsymbol{a},\boldsymbol{b}} = \frac{1}{5!}\left[\sum_{\sigma\in\mathcal{S}_5}\prod_{i=1}^{5}\delta_{a_i b_{\sigma(j)}} - \frac{2}{N+6}\sum_{\substack{\{i_1,i_2,i_3\}\cup\{i_4,i_5\}\\=[\![1,5]\!]}}\ \sum_{\substack{\{j_1,j_2,j_3\}\cup\{j_4,j_5\}\\=[\![1,5]\!]}}\delta_{a_{i_4}a_{i_5}}\delta_{b_{j_4}b_{j_5}}\sum_{\sigma\in\mathcal{S}_3}\prod_{k=1}^{3}\delta_{a_{i_k}b_{j_{\sigma(k)}}}\right.$$

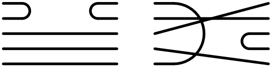
$$\left. + \frac{2}{(N+4)(N+6)}\sum_{\substack{\{i_1\}\cup\{i_2,i_3\}\cup\{i_4,i_5\}\\=[\![1,5]\!]}}\ \sum_{\substack{\{j_1\}\cup\{j_2,j_3\}\cup\{j_4,j_5\}\\=[\![1,5]\!]}}\delta_{a_{i_1}b_{j_1}}\delta_{a_{i_2}a_{i_3}}\delta_{a_{i_4}a_{i_5}}\delta_{b_{j_2}b_{j_3}}\delta_{b_{j_4}b_{j_5}}\right]$$

Types of edges

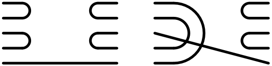
- Unbroken: all strands traverse



- Broken: a pair of corners is connected by a strand at each end of the edge. Rescaled by $1/N$

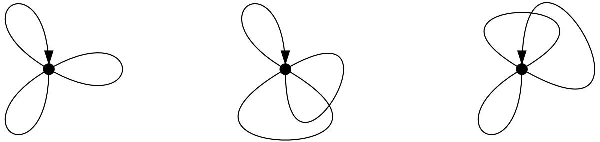


- Doubly broken: two pairs of corners are connected by a strand at each end of the edge. Rescaled by $1/N^2$



Perturbative expansion

- $F_{\boldsymbol{P}}(\lambda)$: sum over rooted connected combinatorial maps $\mathcal{G}$



- Half-edge: represented with five strands
- 945 ways to connect two half-edges
- Projector: combination of those terms with different weights and signs
- Stranded map G: combinatorial map with a choice of one term per edge

$$F_{\boldsymbol{P}}(\lambda) = \sum_{G \text{ connected, rooted}} \lambda^{V(G)} \mathcal{A}(G)$$

$1/N$ expansion

Amplitude of a stranded map:

$$\mathcal{A}(G) = K(G) N^{-\omega(G)} (1 + \mathcal{O}(1/N))$$

- $K(G)$ non-vanishing rational number independent of N
- One free sum = One factor of N per face
- Degree of a stranded map:

$$\omega(G) = 5 + 5V(G) + B(G) + 2B_2(G) - F(G)$$

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Goals:

→ Non-negative degree

→ Maps of zero degree are melonic

Problematic cases

- Simplify the degree by considering number of faces of length p

$$\omega(G) = 5 + B(G) + 2B_2(G) + \sum_p F_p \left(\frac{p}{3} - 1\right)$$

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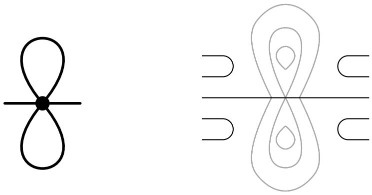
$\rightarrow$ Can be negative iff faces of length $p = 1$ or $p = 2$ (short-faces):

- $p = 1$: Tadpoles, double-tadpoles
- $p = 2$: Melons, dipoles
- Graphs with only long faces: positive degree

Bad double tadpoles



Chain of p double tadpoles:

- 4 faces per vertex
- Factor N^{-5} per vertex
- 2 faces when we glue two double-tadpoles

$$\left(\frac{1}{N}\right)^p N^{2p-1} = N^{p-1}$$

Unbounded from above

$\rightarrow$ non-trivial cancellations

Bounds on combinatorial maps

- Stranded maps with negative degree
- Use irreducibility of the representation to bound the amplitude of the full combinatorial maps
- Double-tadpoles combinatorial maps well-behaved
- Melons: contribute to leading-order

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- Stranded maps with negative degree
- Use irreducibility of the representation to bound the amplitude of the full combinatorial maps
- Double-tadpoles combinatorial maps well-behaved
- Melons: contribute to leading-order
- Problem: generalized double-tadpoles $\rightarrow$ arbitrarily negative degree
- Need to subtract both melons and double-tadpoles

Main theorem

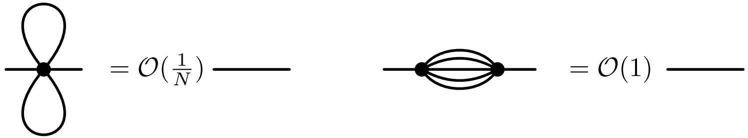
Theorem

We have (in the sense of perturbation series):

$$F_{\boldsymbol{P}}(\lambda) = \sum_{\omega \in \mathbb{N}} N^{-\omega} F_{\boldsymbol{P}}^{(\omega)}(\lambda)\,.$$

- Subtract double-tadpoles and melons
- Restrict to unbroken edges
- Induction: remaining graphs have positive degree

At the combinatorial map level:

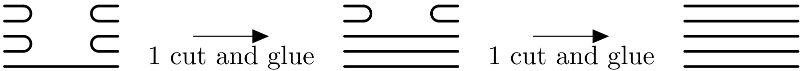


$\Rightarrow$ Crucial role of the irreducibility assumption

- Rewrite out theory with modified covariance and subtracted interaction
- New perturbative expansion in terms of maps $\mathcal{G}$ with no double-tadpoles or melons

Step 2: Restriction to unbroken edges

- Cut and glue: changes the number of faces by -1, 0 or $+1$
- From doubly broken to unbroken propagator:



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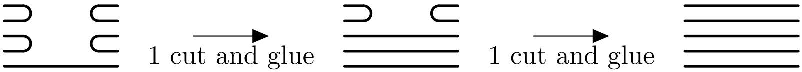

Remove broken edge:

- decrease the number of faces by at most one
- decrease the number of broken edges by one
- the degree can only decrease

Step 3: Remaining graphs

Let G be a stranded graph. If G has no double-tadpole and no melon, then $\omega(G) \geq 0$.

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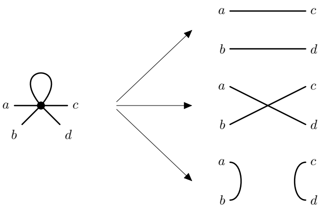
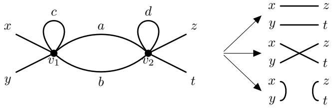
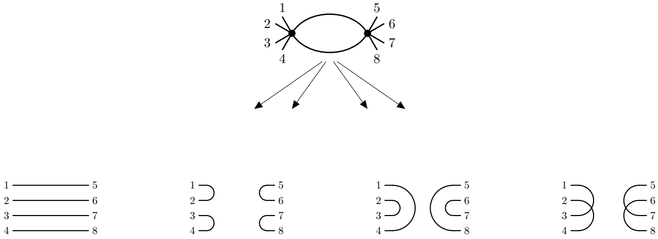
- Look for a strict subgraph that can be deleted without increasing the degree and preserving the constraints
- Exhaustive graph-theoretic distinction of cases
- High number of particular two-point subgraphs to consider

Examples of combinatorial moves







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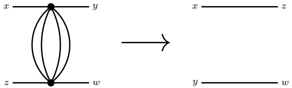
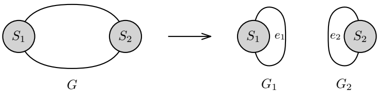
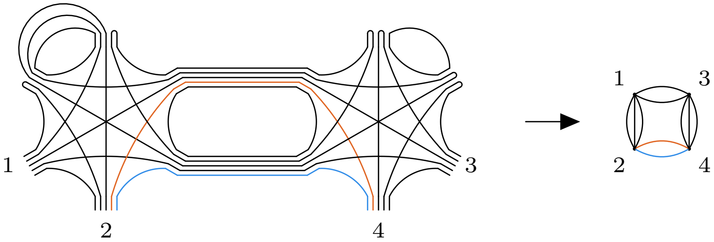
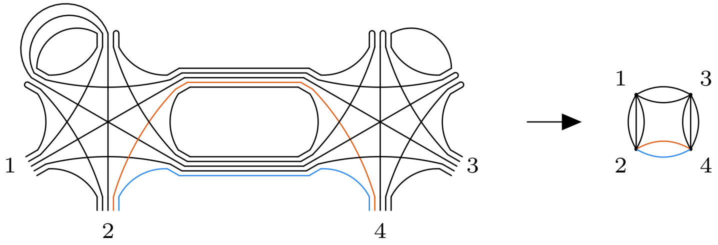


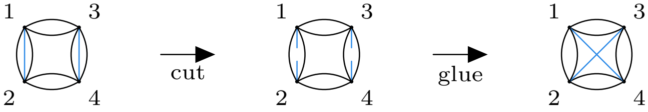


Boundary graphs





- Recursive bounds on $\omega \leftrightarrow$ bounds on flip distance between boundary graphs



End graphs

- Ring graphs ($V = 0$)
- G with no short faces

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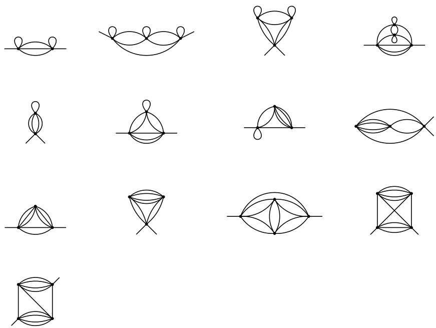
- G with no short faces
- Special cases that need to be treated separately



Leading order

Stranded graphs with no melon can have vanishing degree

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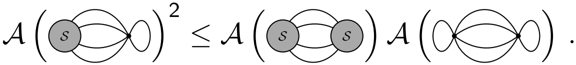
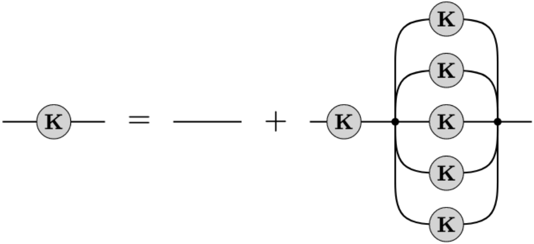
- Bounds on combinatorial maps through Cauchy-Schwarz inequalities



- Maps with no melons are subleading

- **Conclusion: A Feynman map is leading order iff it is melonic**

The two-point function verifies a **closed** SDE



$F_{\boldsymbol{P}}^{(0)}(\lambda)$ is a solution of the polynomial equation:

$$1 - X + m_{\boldsymbol{P}}\lambda^2 X^6 = 0$$

Conclusion and outlook

- Irreducible tensor models with 5-simplex interactions: melonic large-N expansion
- Recursive bounds from a detailed combinatorial analysis of the Feynman graphs.
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Thank you !