Supersymmetric Quantum Mechanics and Morse Theory

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Homework 4: Supersymmetric Quantum Mechanics

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1 Fermionic coherent states

Let us consider a quantum mechanics of complex fermion. Let b and $b^\dagger$ be fermionic creation annihilation operators i.e. they obey anticommutation relation

$$\{b, b^\dagger\} = bb^\dagger + b^\dagger b = 1 \tag{1.1}$$

while the Hilbert space $\mathcal{H}$ is two dimensional with basis

$$|0\rangle, \ \ |1\rangle \equiv b^\dagger|0\rangle, \ \ (-)^F|n\rangle = (-1)^n|n\rangle \tag{1.2}$$

The vacuum state obeys

$$b|0\rangle = 0, \ \ \langle 0|0\rangle = 1. \tag{1.3}$$

We can introduce complex Grassmnann-odd variable ψ and define fermionic coherent states and adjoints

$$|\psi\rangle \equiv e^{b^\dagger\psi}|0\rangle, \quad \langle\bar{\psi}| \equiv \langle 0|e^{\bar{\psi}b} \tag{1.4}$$

1. (*10 Points*) Show that $|\psi\rangle$ are eigenstates of b i.e

$$b|\psi\rangle = \psi|\psi\rangle, \quad \langle\bar{\psi}|b^\dagger = \langle\bar{\psi}|\bar{\psi} \tag{1.5}$$

2. (*10 Points*) Evaluate the paring

$$\langle\bar{\chi}|\psi\rangle \tag{1.6}$$

3. (*10 Points*) Show completeness of the $|\psi\rangle$ basis i.e.

$$1_{\mathcal{H}} = \int d\psi d\bar{\psi}\ e^{\psi\bar{\psi}}|\psi\rangle\langle\bar{\psi}| \tag{1.7}$$

4. (*10 Points*) Prove the trace formula

$$\mathrm{Tr}(\mathcal{O}) = \int d\psi d\bar{\psi}\ e^{\psi\bar{\psi}}\langle -\bar{\psi}|\mathcal{O}|\psi\rangle \tag{1.8}$$

5. (*10 Points*) Prove the supertrace formula

$$\mathrm{Str}(\mathcal{O}) \equiv \mathrm{Tr}((-)^F\mathcal{O}) = \int d\psi d\bar{\psi}\ e^{\psi\bar{\psi}}\langle\bar{\psi}|\mathcal{O}|\psi\rangle \tag{1.9}$$

2 1d Sigma Model

Christoffel symbols Γ for Levi-Cevita connection of the Riemann metric g_{ij} are

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \tag{2.1}$$

The Riemann tensor for the metric g_{ij} can be expressed via

$$R_{abcd} = \frac{1}{2}(\partial^2_{bc} g_{ad} + \partial^2_{ad} g_{bc} - \partial^2_{bd} g_{ac} - \partial^2_{ac} g_{bd}) + g_{ij}(\Gamma^i_{bc}\Gamma^j_{ad} - \Gamma^i_{bd}\Gamma^j_{ac}) \tag{2.2}$$

The $N = 2$ superfield $\hat{x}^i$ in components

$$\hat{x}^j(t, \theta, \bar{\theta}) = x^j(t) + \theta\bar{\psi}^j(t) - \bar{\theta}\psi^j(t) + \theta\bar{\theta}F^j(t) \tag{2.3}$$

with supercovariant derivatives

$$\mathfrak{D} = \frac{\partial}{\partial\theta} - i\bar{\theta}\frac{\partial}{\partial t}, \quad \bar{\mathfrak{D}} = \frac{\partial}{\partial\bar{\theta}} - i\theta\frac{\partial}{\partial t}. \tag{2.4}$$

- (*30 Points*) The superspace realization of sigma-model Lagrangian is given by

$$2L[x, \psi, \bar{\psi}, F] = \int d^2\theta \ \left(g_{jk}(\hat{x})\mathfrak{D}\hat{x}^j\bar{\mathfrak{D}}\hat{x}^k\right) \tag{2.5}$$

  Perform the superspace integration and show that

$$\begin{aligned} 2L[x, \psi, \bar{\psi}, F] = &\, g_{jk}\left(i\bar{\psi}^j\nabla_t\psi^k - i\nabla_t\bar{\psi}^j\psi^k + \dot{x}^j\dot{x}^k\right) \\ &+ g_{lm}(F^l - \psi^k\bar{\psi}^j\Gamma^l_{kj})(F^m - \psi^k\bar{\psi}^j\Gamma^m_{kj}) + \psi^a\bar{\psi}^b\psi^c\bar{\psi}^d R_{abcd} \end{aligned} \tag{2.6}$$

  where

$$\nabla_t\psi^k = \dot{\psi}^k + \Gamma^k_{lm}\dot{x}^l\psi^m \tag{2.7}$$

- (*5 Points*) Describe the SUSY transformations for $x, F, \psi, \bar{\psi}$.

- (*5 Points*) "Integrate out" the auxiliary fields F^i, i.e. solve the equations of motion for F^i as functional of the remaining fields and evaluate the action on this solution.

$$2L[x, \psi, \bar{\psi}] = g_{jk}\left(i\bar{\psi}^j\nabla_t\psi^k - i\nabla_t\bar{\psi}^j\psi^k + \dot{x}^j\dot{x}^k\right) + \psi^a\bar{\psi}^b\psi^c\bar{\psi}^d R_{abcd} \tag{2.8}$$

- (*10 Points*) Describe the SUSY transformations in $x, \psi, \bar{\psi}$ variables.