Supersymmetric Quantum Mechanics and Morse Theory

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Homework 3: $d = 0$ Ssupersymmetry

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1 Linking number

Introduction: For pair of loops (curve with no boundary) C_1 and C_2, given in parametric form

$$C_1 = S^1 \to \mathbb{R}^3 : t \mapsto \gamma_1^i(t), \quad C_2 = S^1 \to \mathbb{R}^3 : t \mapsto \gamma_2^i(t) \tag{1.1}$$

the formula for the linking number is well known

$$\mathrm{lk}(C_1, C_2) = \frac{1}{4\pi} \int_0^{2\pi} dt_1 \int_0^{2\pi} dt_2 \sum_{ijk} \frac{\epsilon_{ijk} \dot{\gamma}_1^i(t_1) \dot{\gamma}_2^j(t_2) (\gamma_1(t_1) - \gamma_2(t_2))^k}{|\gamma_1(t_1) - \gamma_2(t_2)|^3}. \tag{1.2}$$

On the lecture we briefly discussed the relation between the linking number and intersection number. The goal of this problem is to derive the formula above from the intersection theory.

Construction: The Poincare lemma for $\mathbb{R}^3$ implies that

$$H_1(\mathbb{R}^3) = H^1(\mathbb{R}^3) = 0, \tag{1.3}$$

so for any loop $C \subset \mathbb{R}^3$ there exists a surface $S \subset \mathbb{R}^3$ such that

$$\partial S = C. \tag{1.4}$$

For two loops C_1 and C_2 in $\mathbb{R}^3$ we can define linking number via the intersection number

$$\mathrm{lk}(C_1, C_2) = C_1 \cdot S, \tag{1.5}$$

where S is surface, homotopic to a disc, such that $\partial S = C_2$. We can use the integral formula for the intersection number

$$C \cdot S = \int_{\mathbb{R}^3} \eta_C \wedge \eta_S. \tag{1.6}$$

The 1-form η_C is defined via

$$\int_C \omega = \int_{\mathbb{R}^3} \eta_C \wedge \omega, \quad \forall \omega \in \Omega^1(\mathbb{R}^3). \tag{1.7}$$

The Poincare dual form η_C for loop C is trivial in cohomology

$$[\eta_C] = 0 \in H^2(\mathbb{R}^3) = 0, \tag{1.8}$$

hence we can write it as

$$\eta_C = d\eta_S. \tag{1.9}$$

Formally, the 1-form η_S is given by

$$\eta_S = d^{-1}\eta_C, \tag{1.10}$$

but the inverse of external derivative is not a differential operator on forms. We can use homotopy to construct differential operator version of d^{-1}

$$\eta_C = (1 - \Pi_0)\eta_C = (dK + Kd)\eta_C = dK\eta_C \quad \Longrightarrow \eta_S = K\eta_C, \tag{1.11}$$

where homotopy

$$K = \int_0^\infty ds \;\; d^* e^{-s\Delta}. \tag{1.12}$$

1. (*10 points*) Evaluate the Poincare dual η_C for the closed loop given in parametric form

$$C = \gamma : S^1 \to \mathbb{R} : t \mapsto \gamma(t). \tag{1.13}$$

2. (*10 points*) Evaluate the $d^*\eta_C$ for η_C from part 1.

3. (*10 points*) Evaluate the Hodge-Laplacian $\Delta = dd^* + d^*d$ for 1-form on $\mathbb{R}^3$ written in components

$$\omega = \sum_{i=1}^{3} \omega_i(x) dx^i \tag{1.14}$$

   Hint: You can use either of 3 methods:

   - Straightforward evaluation in terms of differential forms.
   - In HW 2 we related d and d^* on $\mathbb{R}^3$ to divergence, gradient and curl, so we can use vector calculus in 3d to evaluate Hodge-Laplacian.
   - In HW 2 we related d and d^* on $\mathbb{R}^3$ to certain differential operators on $\Pi T\mathbb{R}^3 = \mathbb{R}^{3|3}$, so we can use differential operators representation to evaluate Hodge-Laplacian.

4. (*10 points*) Use your answers for $d^*\eta_C$ and Hodge-Laplacian action on forms to evaluate the $\eta_S = K\eta_C$. Use integral representation for δ-function in η_C

$$\delta^3(x) = \prod_{k=1}^{3} \delta(x^k) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} d^3p \; e^{i\sum p_k x^k} \tag{1.15}$$

5. (*10 points*) Evaluate the intersection number

$$C \cdot S = \int_{\mathbb{R}^3} \eta_C \wedge \eta_S \tag{1.16}$$

by performing the $\mathbb{R}^3$ integral first, p-integral second and s-integral last.

2 $d = 0$ $N = 4$ supersymmetry

Let $W(z)$ be the holomorphic ($\bar{z}$-independent) polynomial with complex coefficients. Let us consider a integral over the complex plane

$$Z_W = \frac{1}{2\pi i} \int_{\mathbb{C}} dz d\bar{z} \;\; e^{-W(z)\overline{W(z)}} \partial_z W(z) \overline{\partial_z W(z)} \tag{2.1}$$

We can introduce four Grassmann-odd variables $\psi^z, \bar{\psi}^z,\; \psi^{\bar{z}}$ and $\bar{\psi}^{\bar{z}}$ to rewrite the integral as partition function

$$\begin{aligned} Z_W &= \frac{1}{2\pi i} \int d^4\psi dz d\bar{z} \;\; e^{-S(z,\psi)}, \\ S(z,\psi) &= W(z)\overline{W(z)} - \partial_z W \psi^z \bar{\psi}^z - \overline{\partial_z W(z)} \psi^{\bar{z}} \bar{\psi}^{\bar{z}} \end{aligned} \tag{2.2}$$

1. (*10 points*) Show that the action $S(\psi, z)$ is invariant under the four Grassmann-odd symmetries.

   Hint: You can use one copy of $N = 2$ transformation for z and another copy for $\bar{z}$ so that

$$\begin{aligned} \delta z &= \epsilon^z \bar{\psi}^z + \bar{\epsilon}^z \psi^z \\ \delta \bar{z} &= \epsilon^{\bar{z}} \bar{\psi}^{\bar{z}} + \bar{\epsilon}^{\bar{z}} \psi^{\bar{z}} \end{aligned} \tag{2.3}$$

2. (*10 points*) Show that the integration measure $d^4\psi dz d\bar{z}$ also invariant under the same symmetries (to linear order is symmetry parameters)

3. (*10 points*) An integral of symmetry-invariant action with respect to symmetry-invariant measure localizes. What are the localization points for integral (2.2)?

4. (*10 points*) Evaluate the (Gaussian) integrals around the localization points.

5. (*10 points*) The partition function Z_W is invariant under the shift $W(z) \to W(z) + tf(z)$

with $f(z)$ being holomorphic polynomial with complex coefficients of degree less then degree of W. We can use such invariance to deform the W of degree n to the standard form

$$W(z) = az^n, \quad a \in \mathbb{C} \setminus 0 \tag{2.4}$$

Assuming $W(z)$ has form (2.4) evaluate the integral

$$Z_W = \frac{1}{2\pi i} \int_{\mathbb{C}} dz d\bar{z} \; e^{-W(z)\overline{W(z)}} \partial_z W(z) \overline{\partial_z W(z)} \tag{2.5}$$

Hint: It might be helpful to use radial coordinates r, ϕ on $\mathbb{C}$

$$z = re^{i\phi}, \quad \bar{z} = re^{-i\phi}, \quad r \in [0, +\infty), \quad \phi \in [0, 2\pi). \tag{2.6}$$