Spinors and SUSY on a worldline

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1 Introduction

I was invited to give a two lectures as a part of the Special Relativity course at OIST. Among many topics the course included the world-line description of the scalar particle and spinors in four-dimensional Minkowski. So I decided to discuss the world-line descriptions of the spacetime spinors. Below is the literature that I used to prepare for my lectures

1. Link [1] to the SR course website in OIST.
2. Classical book by Polyakov [2] has a nice introduction to the relativistic particle, world-line gravity and supergravity.
3. Nice article [3] with careful description of the worldline supersymmetry, multiplets and external field couplings.
4. Nice lectures [4] on the worldline formalism including amplitudes and schwinger process in details.
5. An interesting discussion [5] of the quantum particle on a curved background.
6. Nice PHD thesis [6] on Worldline approach to the higher spins.

2 Spinors via Quantization of the Grassmann Symplectic manifold

We want to construct a world-line description for the 4d particles with a spin. In case of the scalar particles the the wave functions for the QM of the world-line particle were scalars $\phi(x)$ and transformed trivially under the Lorentz group. In case of particles with spin in spacetime we want the corresponding Hilbert space to contain a finite number functions, that describe the multiple components of the spinor

$$\Psi^{\alpha}(x) = \lambda^{\alpha} e^{ix^{\beta\dot{\beta}}\lambda_{\beta}\lambda_{\dot{\beta}}}. \tag{2.1}$$

Let us start with the world-line description of the finite-dimensional space of polarization (helicities). During previous lectures we discussed that the spinor representations of the $SO(1,3)$ algebra can be described by the Clifford algebra

$$\{\gamma^{\mu}, \gamma^{\nu}\} = \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\eta^{\mu\nu} \tag{2.2}$$

The algebra above is the same as the canonical quantization in $\hbar = 1$ convention of the phase space with coordinates ψ^μ and Poisson bracket

$$\{\psi^\mu, \psi^\nu\}_{pb} = -2i\eta^{\mu\nu}. \tag{2.3}$$

The Poisson bracket for the homogeneous functions A, B is defined as

$$\{A, B\}_{pb} = -i\eta^{\mu\nu}\left(\frac{\partial A}{\partial\psi^\mu}\frac{\partial B}{\partial\psi^\nu} - (-1)^{|A||B|}\frac{\partial B}{\partial\psi^\mu}\frac{\partial A}{\partial\psi^\nu}\right), \tag{2.4}$$

with $|A|, |B| \in \{0, 1\}$ being parities of and A, B. We can describe such Poisson bracket by a simplectic form

$$\omega = i\eta^{\mu\nu}\delta\psi_\mu \wedge \delta\psi_\nu \tag{2.5}$$

on symplectic manifold with coordinates ψ^μ. The classical limit $\hbar \to 0$ of the Clifford algebra

$$\hat{\psi}^\mu\hat{\psi}^\nu + \hat{\psi}^\nu\hat{\psi}^\mu = 2\hbar\eta^{\mu\nu} \to 0 \tag{2.6}$$

tells us that the coordinates ψ^μ are anticommutting or Grassmann variables

$$\psi^\mu\psi^\nu = -\psi^\nu\psi^\nu. \tag{2.7}$$

Furthermore we can be obtain our symplectic form from the first order action

$$S = \int \frac{i}{4}\eta_{\mu\nu}\psi^\mu\dot{\psi}^\nu d\tau. \tag{2.8}$$

Let us get some understanding of the Grassmann variables. Let us start with the case of a single Grassmann variable ψ. The simplest object is the functions on ψ, that we can treat as the Taylor series

$$f(\psi) = f_0 + f_1\psi + f_2\psi^2 + ... = f_0 + f_1\psi, \;\; \psi^2 = 0. \tag{2.9}$$

Such series are always finite and we do not need to worry about the convergence issues. We will define *parity* of the individual terms as the total number of Grassmann variables mod 2

$$|\psi| = 1, \;\; |\psi_1\psi_2| = 0, \;\; |\psi_1\psi_2\psi_3| = 1, ... \tag{2.10}$$

Similarly we can define derivatives as the sum of derivatives, acting on the individual terms in Taylor series

$$\partial_\psi f(\psi) = f_0 \partial_\psi (1) + f_1 \partial_\psi (\psi) = f_1. \tag{2.11}$$

Derivative and multiplication by ψ obey

$$\{\partial_\psi, \psi\} = 1, \tag{2.12}$$

that can be checked directly

$$\{\partial_\psi, \psi\} f(\psi) = \partial_\psi (\psi f(\psi)) + \psi \partial_\psi f(\psi) = \partial_\psi (f_0 \psi) + \psi f_1 = f_0 + f_1 \psi = f(\psi). \tag{2.13}$$

In case of multiple variables it generalizes to

$$\{\partial_{\psi^i}, \psi^j\} = \delta^j{}_i, \quad \{\partial_{\psi_i}, \partial_{\psi_j}\} = 0. \tag{2.14}$$

We can use the relation above to describe the Hilbert space of our QM (2.8) or equivalently the representations of Clifford algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \tag{2.15}$$

in terms of the functions over the half of the phase space variables. Let us define

$$\gamma_1^\pm = \frac{1}{2}(\gamma_1 \pm \gamma_0), \quad \gamma_2^\pm = \frac{1}{2}(\gamma_3 \pm i\gamma_2), \tag{2.16}$$

so that

$$\{\gamma_i^+, \gamma_j^-\} = \delta_{ij}, \quad \{\gamma_i^+, \gamma_j^+\} = \{\gamma_i^-, \gamma_j^-\} = 0. \tag{2.17}$$

Note: We can use $\gamma_i^\pm$ to describe the symplectic structure and the fermionic action since they provide a complete decomposition into positions γ_i^- and momenta γ_i^+ The Poisson bracket and symplectic structure

$$\{\psi_i^+, \psi_j^-\} = \delta_{ij}, \ \Rightarrow \{\psi_i^+, \psi_j^-\}_{pb} = -i\delta_{ij}, \ \Rightarrow \ \omega = i\delta^{ij} \delta\psi_i^+ \delta\psi_j^-. \tag{2.18}$$

While the Lagrangian becomes

$$L = i\delta^{ij} \psi_i^+ \dot{\psi}_j^- = \frac{i}{4}(\psi_1 + \psi_0)(\dot{\psi}_1 - \dot{\psi}_0) + \frac{i}{4}(\psi_3 + i\psi_2)(\dot{\psi}_3 - i\dot{\psi}_2). \tag{2.19}$$

Using the Minkowski metric $\eta_{\mu\nu}$

$$L = \frac{i}{4}\eta_{\mu\nu}\psi^\mu\dot{\psi}^\nu + \partial_\tau(...). \tag{2.20}$$

We can represent the $\gamma_i^\pm$ as

$$\gamma_i^+ = \theta_i, \quad \gamma_i^- = \frac{\partial}{\partial\theta_i} \tag{2.21}$$

acting on functions $f(\theta_1, \theta_2)$. The space of such functions is four dimensional

$$f(\theta_1, \theta_2) = f_{00} + f_{01}\theta_1 + f_{10}\theta_2 + f_{11}\theta_1\theta_2, \tag{2.22}$$

thus we can always write $\gamma_i^\pm$ as a 4×4 matrices. The 4d representation of the Clifford algebra in 4d is the same as Dirac spinors. Previously we have learned that the smallest spinor representation in 4d is the 2-component Weyl spinors. So Dirac spinor is reducible representation that consists of two Weyl spinors. Let us see how the same logic follows from our QM. The Lorentz generators $L_{\mu\nu}$ are related to the Clifford algebra generators via

$$L_{\mu\nu} = -\frac{i}{4}[\gamma_\mu, \gamma_\nu], \tag{2.23}$$

that can be verified by the simple check

$$i[L_{\mu\nu}, L_{\lambda\rho}] = \eta_{\mu\lambda}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\lambda} - \eta_{\nu\lambda}L_{\mu\rho} + \eta_{\nu\rho}L_{\mu\lambda}. \tag{2.24}$$

Lorentz generators are quadratic in γ^μ so they preserve the parity! To formalize this statement let us introduce the γ_5 as

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3, \quad \gamma_5^2 = 1, \quad \{\gamma_5, \gamma_\mu\} = 0, \tag{2.25}$$

so the parity $|A|$ is

$$\gamma_5 A = (-1)^{|A|}A. \tag{2.26}$$

Lorentz generators commute with γ_5

$$[L_{\mu\nu}, \gamma_5] = 0 \tag{2.27}$$

so the irreducible Clifford algebra representations are reducible for the Lorentz group. In particular the 4d Dirac spinor representation reduces to the pair of Weyl spinor representations of the odd and even functions $f(\theta_1, \theta_2)$.

Question: What the inner-product on the Hilbert space? The naive guess for the inner-product would be to use

$$\langle f, g\rangle = \int d^2\theta f(\theta) g^*(\theta) \tag{2.28}$$

while we need to supplement it with some a complex conjugation that turns out to be of the following form

$$\theta^* = \theta, \quad (\theta_1\theta_2...\theta_n)^* = \theta_n\theta_{n-1}...\theta_1 \tag{2.29}$$

The integration over Grassmann variables is performed using the following relation

$$\int f(\theta) d\theta = \int (f_0 + f_1\theta) d\theta = f_1 \int \theta d\theta = f_1 \tag{2.30}$$

in case of multiple variables the integration picks up the highest power term form the function i.e.

$$\int f(\theta_i) \prod d\theta_i = \int f_{11...11}\theta_1...\theta_n \prod d\theta_i = f_{11...11} = \frac{\partial^n f(\theta_i)}{\partial\theta_1...\partial\theta_n} \tag{2.31}$$

In case of the Weyl spinors with $\gamma_5 = -1$ we have

$$\chi = \chi^\alpha\theta_\alpha, \quad \lambda = \lambda^\alpha\theta_\alpha \tag{2.32}$$

and

$$\begin{aligned} \langle\chi, \lambda\rangle &= \int d^2\theta \chi(\theta)\lambda^*(\theta) = \int d^2\theta(\chi^1\theta_1 + \chi^2\theta_2)(\lambda^1\theta_1 + \lambda_2\theta_2) \\ &= \int d\theta_1 d\theta_2(\chi^1\lambda^2 - \chi^2\lambda^1)\theta_1\theta_2 = \epsilon_{\alpha\beta}\chi^\alpha\lambda^\beta, \quad \epsilon_{12} = 1 \end{aligned} \tag{2.33}$$

While the Weyl spinors with $\gamma_5 = +1$ we have

$$\chi = \chi^{\dot{1}} + \chi^{\dot{2}}\theta_1\theta_2, \quad \lambda = \lambda^{\dot{1}} + \lambda^{\dot{2}}\theta_1\theta_2 \tag{2.34}$$

and

$$\begin{aligned} \langle\chi, \lambda\rangle &= \int d^2\theta \chi(\theta)\lambda^*(\theta) = \int d^2\theta(\chi^{\dot{1}} + \chi^{\dot{2}}\theta_1\theta_2)(\lambda^{\dot{1}} - \lambda^{\dot{2}}\theta_1\theta_2) \\ &= \int d\theta_1 d\theta_2(\chi^{\dot{2}}\lambda^{\dot{1}} - \chi^{\dot{1}}\lambda^{\dot{2}})\theta_1\theta_2 = \epsilon_{\dot{\alpha}\dot{\beta}}\chi^{\dot{\alpha}}\lambda^{\dot{\beta}}, \quad \epsilon_{\dot{1}\dot{2}} = -1. \end{aligned} \tag{2.35}$$

3 $N = 1$ Superparticle

Let us combine both the fermionic and bosonic actions

$$S[x, \psi] = \int \left(p_\mu \dot{x}^\mu + \frac{i}{4}\psi_\mu\dot{\psi}^\mu \right) d\tau. \tag{3.1}$$

The action above admits a lot of conserved quantities, in particular there are two types of Lorentz transformations: the one is action on the world-line fermions with generators

$$S_{\mu\nu} = \frac{i}{2}\psi_\mu\psi_\nu \tag{3.2}$$

and the other one acting on bosonic variables

$$L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu. \tag{3.3}$$

As we discussed in previous section the wave-functions for the QM with action (3.1) are space-time spinors Ψ^α that depend on position x^μ. The two Lorentz groups act as

$$\Psi_\alpha(x) \to R(\Lambda_1)^\beta{}_\alpha \Psi_\beta(\Lambda_2^{-1}x), \ \ {\Lambda_1}^\mu{}_\nu, {\Lambda_2}^\mu{}_\nu \in SO(1,3), \tag{3.4}$$

with R being the spinor representation (the $SU(2)$ matrices that we discussed last lecture). The spinor field in a space-time transforms with respect to the Lorentz group via

$$\Psi_\alpha(x) \to R(\Lambda)^\beta{}_\alpha \Psi_\beta(\Lambda^{-1}x), \ \ \Lambda^\mu{}_\nu \in SO(1,3). \tag{3.5}$$

We want to design our particle so that the Lorentz generators match on the physical states

$$L_{\mu\nu} - 2S^{\mu\nu} \approx 0. \tag{3.6}$$

The factor of 2 due to the spin$-1/2$ nature of the space-time field. Remember the π rotation for the spinor while doing 2π rotation of the vector x^μ. One way to achieve this is to introduce a local (gauge) symmetry Q so that

$$L_{\mu\nu} - 2S^{\mu\nu} = Q(...). \tag{3.7}$$

So on the states that are annihilated by Q angular momentum operators are equal. The Q operator should be compatible with dynamics i.e.

$$[Q, H] = \partial_\tau Q = 0. \tag{3.8}$$

From the equality (3.7) we conclude that Q should mix ψ^μ and x^μ, p_μ variables. Given our Lagrangian structure (3.1) we can deduce that Q is like a rotation in $x^\mu \to \psi^\mu \to p^\mu$

$$\delta_\epsilon x^\mu = i\epsilon\psi^\mu, \quad \delta_\epsilon\psi^\mu = -2\epsilon p^\mu, \quad \delta p_\mu = 0. \tag{3.9}$$

The $\epsilon(\tau)$ is an Grassmanian parameter for the transformation.
Check

$$\begin{gathered}\delta\left(p_\mu\dot{x}^\mu + \tfrac{i}{4}\psi_\mu\dot{\psi}^\mu\right) = \delta p_\mu\dot{x}^\mu + p_\mu\delta\dot{x}^\mu + \tfrac{i}{4}\delta\psi_\mu\dot{\psi}^\mu + \tfrac{i}{4}\psi_\mu\delta\dot{\psi}^\mu \\ = ip_\mu\partial_\tau(\epsilon\psi^\mu) - \tfrac{i}{4}2\epsilon p_\mu\dot{\psi}^\mu - \tfrac{i}{4}\psi_\mu 2\partial_\tau(\epsilon p^\mu) = i\dot{\epsilon}p_\mu\psi^\mu - \tfrac{i}{2}\dot{\psi}^\mu\epsilon p_\mu - \tfrac{i}{2}\psi_\mu\partial_\tau(\epsilon p^\mu) \\ \boxed{\delta_\epsilon L = i\dot{\epsilon}p_\mu\psi^\mu - \frac{i}{2}\partial_\tau(\epsilon\psi_\mu p^\mu)}\end{gathered} \tag{3.10}$$

According to the Noether's theorem the symmetries are generated by the charges on a phase space

$$\delta_\epsilon F(x,p,\psi) = \{\epsilon Q, F\}_{pb}, \quad Q = i\psi^\mu p_\mu. \tag{3.11}$$

In particular

$$\{Q, x^\mu\}_{pb} = i\psi^\mu, \quad \{Q,\psi^\mu\}_{pb} = -2p^\mu, \quad \{Q, p^\mu\}_{pb} = 0. \tag{3.12}$$

The Lorentz generator matching (3.7) becomes

$$L_{\mu\nu} - 2S^{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu - i\psi_\mu\psi_\nu = -\frac{1}{2}Q(x_\mu\psi_\nu - x_\nu\psi_\mu). \tag{3.13}$$

3.1 Local SUSY

Our trivial action (3.1) is invariant under the *global* supersymmetry, i.e. transformation parameter ϵ is a constant. While we want to achieve the *local* susy with $\epsilon = \epsilon(t)$. It is possible if we introduce an additional fermionic field - gravitino χ and include the relativistic Hamiltonian. The full action becomes

$$S = \int\left(p_\mu\dot{x}^\mu + \frac{i}{4}\psi_\mu\dot{\psi}^\mu - ep^2 - i\chi\psi^\mu p_\mu\right)d\tau. \tag{3.14}$$

The SUSY transformations for the gravity multiplet

$$\delta_\epsilon e = 2i\epsilon\chi, \quad \delta_\epsilon\chi = \dot{\epsilon}. \tag{3.15}$$

The additional terms transforms via

$$\begin{aligned}\delta(-ep^2 - i\chi\psi^\mu p_\mu) &= -\delta e p^2 - i\delta\chi\psi^\mu p_\mu - i\chi\delta\psi^\mu p_\mu \\ &= i\cancel{2\epsilon\chi p^2} - i\dot{\epsilon}\psi^\mu p_\mu + i\cancel{2\chi\epsilon p^\mu p_\mu} = -i\dot{\epsilon}\psi^\mu p_\mu,\end{aligned} \tag{3.16}$$

so that the full action is invariant

$$\delta S = -\frac{i}{2}\int d\tau \partial_\tau(\epsilon\psi_\mu p^\mu). \tag{3.17}$$

The action for our superparticle (3.14) is a linear combination of the two constraints

$$H = p^2, \quad Q = \psi^\mu p_\mu, \tag{3.18}$$

that form the $N = 1$ superalgebra

$$\{Q, Q\} = 2Q^2 = 2H, \quad [Q, H] = 0. \tag{3.19}$$

In a canonical quantization

$$p_\mu \to -i\partial_\mu, \quad \psi^\mu \to \gamma^\mu, \tag{3.20}$$

so the supercharge constraint become a Dirac operator

$$-i\gamma^\mu\partial_\mu\Psi(x) = 0. \tag{3.21}$$

3.2 Cosmological term

Massive relativistic particle is described by the following world line action

$$S = \int d\tau \left(p_\mu \dot{x}^\mu - e p_\mu p^\mu - e m^2\right) = S_{m=0} - m^2 S_\Lambda, \tag{3.22}$$

with

$$S_\Lambda = \int_0^1 e \, d\tau. \tag{3.23}$$

So effectively the mass of the relativistic particle is a coupling for the 1D cosmological term. Since we want to have our action to obey 1D local supersymmetry we need to proper generalize the cosmological term. Polyakov in his book provides an observation that the correct cosmological term corresponds to the invariant distance in superspace. Since distance is bosonic we can try the following ansatz

$$S_\Lambda = \int e d\tau + \int d\tau_1 \int d\tau_2 \; K(\tau_1, \tau_2)\chi(\tau_1)\chi(\tau_2) \tag{3.24}$$

We can further restrict kernel K using reparametization and SUSY invariance. In fact we do not need to consider 1D Diffeos since they contained inside local SUSY as commutators! Local SUSY

$$\delta_\epsilon e = 2i\epsilon\chi, \quad \delta_\epsilon \chi = \dot{\epsilon} \tag{3.25}$$

invariance leads to the following relation

$$0 = \delta_\epsilon S_\Lambda = \int 2i\epsilon\chi \; d\tau + \int d\tau_1 \int d\tau_2 \; K(\tau_1, \tau_2)\dot{\epsilon}(\tau_1)\chi(\tau_2) + \int d\tau_1 \int d\tau_2 \; K(\tau_1, \tau_2)\chi(\tau_1)\dot{\epsilon}(\tau_2),$$

$$2i\delta(\tau_1 - \tau_2) - \frac{\partial K(\tau_1, \tau_2)}{\partial \tau_1} + \frac{\partial K(\tau_1, \tau_2)}{\partial \tau_2} = 0. \tag{3.26}$$

Thus we conclude that

$$K(\tau_1, \tau_2) = \frac{i}{2}\mathrm{sign}(\tau_1 - \tau_2), \quad \mathrm{sign}(\pm 0) = \pm 1. \tag{3.27}$$

The SUSY invariant distance becomes

$$S_\Lambda = \int e(\tau) d\tau + \frac{i}{2} \int d\tau_1 \int d\tau_2 \; \mathrm{sign}(\tau_1 - \tau_2)\chi(\tau_1)\chi(\tau_2). \tag{3.28}$$

The action S_Λ is nonlocal, but we can write it as a local expression is we introduce an extra field ψ_5 so that

$$m^2 S_\Lambda = \int d\tau \quad \left(m^2 e - \frac{i}{4}\psi_5\dot{\psi}_5 + im\psi_5\chi \right), \tag{3.29}$$

while ψ_5 obeys the classical equations of motion

$$-\frac{i}{2}\dot{\psi}_5 + im\chi = 0 \quad \psi_5 = 2m \int d\tau \chi(\tau). \tag{3.30}$$

The SUSY transformation of the new field

$$\delta\psi_5 = -2m\epsilon(t), \quad \delta e = 2i\epsilon\chi, \quad \delta\chi = \dot{\epsilon} \tag{3.31}$$

Check:

$$\delta_\epsilon\left(m^2 e - \tfrac{i}{4}\psi_5\dot{\psi}_5 + im\psi_5\chi\right) = \cancel{2im^2\epsilon\chi} + \tfrac{i}{2}m\epsilon\dot{\psi}_5 + \tfrac{i}{2}m\psi_5\dot{\epsilon} - \cancel{2im^2\epsilon\chi} - mi\psi_5\dot{\epsilon} = \tfrac{i}{2}\partial_t(m\epsilon\psi_5). \tag{3.32}$$

The full action for the massive superparticle particle is of the form

$$S = S_{m=0} - m^2 S_\Lambda = \int\left(p_\mu\dot{x}^\mu + \frac{i}{4}\psi_\mu\dot{\psi}^\mu + \frac{i}{4}\psi_5\dot{\psi}_5 - e(m^2+p^2) - i\chi(\psi^\mu p_\mu - m\psi_5)\right)d\tau. \tag{3.33}$$

The constraints

$$-\frac{\partial L}{\partial e} = H = p^2 + m^2, \quad i\frac{\partial L}{\partial \chi} = Q = \psi^\mu p_\mu - m\psi_5 \tag{3.34}$$

satisfy the $N=1$ superalgebra

$$\{Q,Q\}_{pb} = 2H, \quad \{Q,H\}_{pb} = 0. \tag{3.35}$$

The canonical quantization of the symplectic structure in our action imposes the (anti)commutation relations for the operator $\hat{\psi}_5$ of the form

$$\{\hat{\psi}_5,\hat{\psi}_5\} = 2, \quad \{\hat{\psi}_5,\hat{\psi}_\mu\} = 0, \tag{3.36}$$

which are the same as the commutation relation for the $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. So we can represent the $\hat{\psi}_5$ in the Hilbert space that we used in massless case. The Dirac equation becomes

$$(\hat{\psi}^\mu\hat{p}_\mu - m\hat{\psi}_5)|\Psi\rangle = 0 \;\Rightarrow i\gamma^\mu\partial_\mu\Psi = m\gamma_5\Psi. \tag{3.37}$$

Using the Dirac spinor decomposition into the left and right Weyl spinors

$$\Psi = \chi + \lambda, \quad \gamma^5\Psi = \lambda - \chi, \tag{3.38}$$

the massive Dirac equation becomes

$$i\sigma^\mu\partial_\mu\chi = m\lambda, \quad i\sigma^\mu\partial_\mu\lambda = m\chi, \tag{3.39}$$

which is exactly the pair of equations that we discussed on previous lectures using the geometric approach to spinors.

4 Superparticle in with charge

In previous lectures we discussed the scalar particle coupling to the Maxwell field

$$S[A] = \int d\tau \left(\frac{1}{2}\dot{x}_\mu^2 + q\dot{x}^\mu A_\mu \right). \tag{4.1}$$

The gauge invariance is trivial and does not require equations of motion for x^μ

$$\delta S[A] = \int d\tau \left(\dot{x}^\mu \partial_\mu \epsilon \right) = \int_0^T d\tau \; \partial_\tau \epsilon. \tag{4.2}$$

We want to add some fermionic terms to make the interactions compatible with (global) SUSY in position space

$$\delta_\epsilon x = \epsilon\psi, \quad \delta_\epsilon \psi = \epsilon\dot{x}. \tag{4.3}$$

Note: From this point we will use different SUSY transformations. Previous transformations differ by certain numerical factors that are combinations of $\pm 1, i, 2$. The massless SUSY particle is described by

$$\int d\tau \left(\frac{1}{2}\dot{x}_\mu^2 + \frac{1}{2}\dot{\psi}^\mu \psi_\mu \right). \tag{4.4}$$

The ansatz for the invariant action would be

$$S[A] = q \int d\tau \left(\dot{x}^\mu A_\mu + \alpha F_{\mu\nu}\psi^\mu\psi^\nu \right), \tag{4.5}$$

while α is fixed by the SUSY invariance

$$\begin{aligned} \delta L = \delta\left(\dot{x}^\mu A_\mu + \alpha F_{\mu\nu}\psi^\mu\psi^\nu\right) &= \delta\dot{x}^\mu A_\mu + \dot{x}^\mu \delta A_\mu + \alpha\delta F_{\mu\nu}\psi^\mu\psi^\nu + \alpha F_{\mu\nu}\delta\psi^\mu\psi^\nu + \alpha F_{\mu\nu}\psi^\mu\delta\psi^\nu \\ &= \partial_\tau(\epsilon\psi^\mu)A_\mu + \dot{x}^\mu\epsilon\psi^\nu\partial_\nu A_\mu + \alpha\epsilon\psi^\lambda\partial_\lambda F_{\mu\nu}\psi^\mu\psi^\nu + \alpha F_{\mu\nu}\epsilon\dot{x}^\mu\psi^\nu + \alpha F_{\mu\nu}\psi^\mu\epsilon\dot{x}^\nu \\ &= \partial_\tau(\epsilon\psi^\mu)A_\mu + \epsilon\psi^\nu\partial_\tau A_\nu + \dot{x}^\mu\epsilon\psi^\nu(\partial_\nu A_\mu - \partial_\mu A_\nu) + 2\alpha F_{\mu\nu}\epsilon\dot{x}^\mu\psi^\nu \\ &\boxed{\delta L = \partial_\tau(\epsilon\psi^\mu A_\mu) + (2\alpha - 1)F_{\mu\nu}\epsilon\dot{x}^\mu\psi^\nu} \end{aligned} \tag{4.6}$$

where we used

$$F_{\mu\nu} = -F_{\nu\mu}, \quad \partial_{[\lambda}F_{\mu\nu]} = 0 \tag{4.7}$$

So we have a EM interaction that is invariant under the global SUSY

$$S[A] = \int d\tau \left(\frac{1}{2}\dot{x}_\mu^2 + \frac{1}{2}\dot{\psi}^\mu\psi_\mu + q\dot{x}^\mu A_\mu + \frac{q}{2}F_{\mu\nu}\psi^\mu\psi^\nu \right), \tag{4.8}$$

We want to have local SUSY so we need to introduce the gauge multiplet e, χ and follow the same construction as we did for the free massless particle. However we can notice an interesting structure of the SUGRA action in massless free case

$$S[e,\chi] = S_{top} + \int eH + \int \chi Q, \tag{4.9}$$

with

$$S_{top} = \frac{1}{2}\int (p_\mu \dot{x}^\mu + \psi_\mu \dot{\psi}^\mu) d\tau \tag{4.10}$$

being a symplectic structure part of the action that is manifestly invariant under the 1d Diffs, therefore called topological as being independent on $e(\tau)$. The local SUSY transformation

$$\delta S_{top} = \int \dot{\epsilon}(\tau) Q + \int \partial_\tau (....), \quad Q = \psi^\mu p_\mu \tag{4.11}$$

can be fixed if we add the gravitino coupling of the form

$$S[\chi] = S_{top} + \int \chi Q, \quad \delta\chi = -\dot{\epsilon} \tag{4.12}$$

The action above is still not quite local SUSY invariant since

$$\delta S[\chi] = S_{top} + \int \chi \delta_\epsilon Q, \quad \delta_\epsilon Q = \{Q, \epsilon Q\}_{pb} = 2\epsilon H \tag{4.13}$$

that we can fix by adding metric coupling

$$S[e,\chi] = S_{top} + \int \chi Q + \int eH, \quad \delta e = 2\epsilon\chi \tag{4.14}$$

Fortunately, our iterative construction end at this step since

$$\delta(eH) = \delta e H + e\delta H = 2\epsilon\chi H + e\{H, Q\}_{pb} = -\chi\delta_\epsilon Q. \tag{4.15}$$

In case of the presence of gauge field A the topological action is modified

$$S_{top} = \int \left(\frac{1}{2} p_\mu \dot{x}^\mu + \frac{1}{2}\psi_\mu \dot{\psi}^\mu + qA_\mu \dot{x}^\mu \right) \tag{4.16}$$

so the global SUSY is generated by modified charge

$$Q = \psi^\mu (p_\mu - qA_\mu) \tag{4.17}$$

According to the local SUSY invariant action construction the hamiltonian becomes

$$2H = \{Q, Q\}_{pb} = (p_\mu - qA_\mu(x))^2 - q\psi^\mu\psi^\nu F_{\mu\nu}(x). \tag{4.18}$$

The Dirac equation is modified into

$$Q|\Psi\rangle = 0 \quad \Rightarrow \gamma^\mu(i\partial_\mu - qA_\mu)\Psi = 0. \tag{4.19}$$

Let us look at some features of the SUSY hamiltonian (4.18)

- In absence of the grassmann coordinates ψ^μ it is the same as the Hamiltonian for the scalar particle coupled to Maxwell field.
- The last term describes the spin coupling to the field since $\psi^\mu\psi^\nu$ is the spin generator $S^{\mu\nu}$ acting on the fermionic part of the Hilbert space.

5 Superspace

The $N = 1$ superspace has two coordinates: world-line time t and grassmanian coordinate θ. The supersymmetry acts via

$$\begin{aligned} &\delta_\epsilon t = -\epsilon\theta, \quad \delta_\epsilon\theta = \epsilon, \\ &\epsilon^2 = 0, \quad \epsilon\theta + \theta\epsilon = 0. \end{aligned} \tag{5.1}$$

The commutator for pair of such transformations

$$\begin{gathered} [\delta_2, \delta_1]t = \delta_2(-\epsilon_1\theta) - \delta_1(-\epsilon_2\theta) = -\epsilon_1\epsilon_2 + \epsilon_2\epsilon_1 = 2\epsilon_2\epsilon_1, \\ [\delta_2, \delta_1]\theta = 0 \end{gathered} \tag{5.2}$$

The generator of such transformations

$$\epsilon Q = \delta_\epsilon t\frac{\partial}{\partial t} + \delta_\epsilon\theta\frac{\partial}{\partial\theta} = \epsilon\left(\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial t}\right), \quad Q^2 = -\partial_t \tag{5.3}$$

It is useful to introduce a supercovariant derivative $\mathcal{D}$ defined to be anticommutting with Q

$$\{\mathcal{D}, Q\} = 0, \quad \mathcal{D} = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial t}, \;\; \mathcal{D}^2 = \partial_t. \tag{5.4}$$

The SUSY action on superspace is naturally extended to SUSY action on the functions on superspace $\hat{x}(t,\theta)$ often called superfields. Any such function is a finite Taylor series in θ

$$\hat{x}(t,\theta) = x(t) + \theta\psi(t), \tag{5.5}$$

while the supersymmetry acts via

$$\begin{gathered} \delta_\epsilon \hat{x} = \epsilon Q(\hat{x}) = \epsilon\psi(t) - \epsilon\theta\dot{x}(t) \\ \delta_\epsilon x = \epsilon\psi, \quad \delta_\epsilon\psi = \epsilon\dot{x}. \end{gathered} \tag{5.6}$$

Using the supercovariant derivatives we can construct manifestly supersymmetric actions. In particular the following expression

$$S = \int dt d\theta F(\hat{x}, \mathcal{D}\hat{x}, ..., \mathcal{D}^n\hat{x}, ...) \tag{5.7}$$

is SUSY invariant since

$$\delta S = \int dt d\theta \delta F = \int dt d\theta Q(F) = \int dt d\theta \left(\frac{\partial F}{\partial \theta} - \theta\frac{\partial F}{\partial t}\right) = 0 - \int dt\ \partial_t F \tag{5.8}$$

where we used supercovariant derivatives definition

$$\delta(\mathcal{D}\hat{x}) = \mathcal{D}\delta\hat{x} = \mathcal{D}Q(\hat{x}) = Q(\mathcal{D}\hat{x}), \tag{5.9}$$

and Grassmann variable integration

$$\int d\theta \frac{\partial F}{\partial \theta} = 0. \tag{5.10}$$

Example 1: The free massless superparticle action is a superspace integral of the form

$$S = \frac{1}{2}\int dt d\theta\ \mathcal{D}^2\hat{x}^\mu\mathcal{D}\hat{x}_\mu = \frac{1}{2}\int dt d\theta\ (\dot{x}^\mu + \theta\dot{\psi}^\mu)(\psi_\mu + \theta\dot{x}_\mu) = \frac{1}{2}\int dt\ (\dot{x}_\mu^2 + \dot{\psi}^\mu\psi_\mu) \tag{5.11}$$

The Legendre transform of the action above

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \dot{x}_\mu, \quad S = \frac{1}{2}\int d\tau \left(2p_\mu\dot{x}^\mu + \dot{\psi}^\mu\psi_\mu + p_\mu p^\mu\right) \tag{5.12}$$

is the superparticle action (3.14) in the gauge $e = 1, \chi = 0$.

Example 2: The Maxwell field interaction is realized by the superspace integral

$$\int dt d\theta A_\mu(\hat{x}) \mathcal{D}\hat{x}^\mu = \int dt d\theta \, (A_\mu(x) + \theta\psi^\nu\partial_\nu A_\mu(x))(\psi_\mu + \theta\dot{x}_\mu) = \int dt \left(A_\mu \dot{x}^\mu - \frac{1}{2}\psi^\mu\psi^\nu F_{\mu\nu} \right). \tag{5.13}$$

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