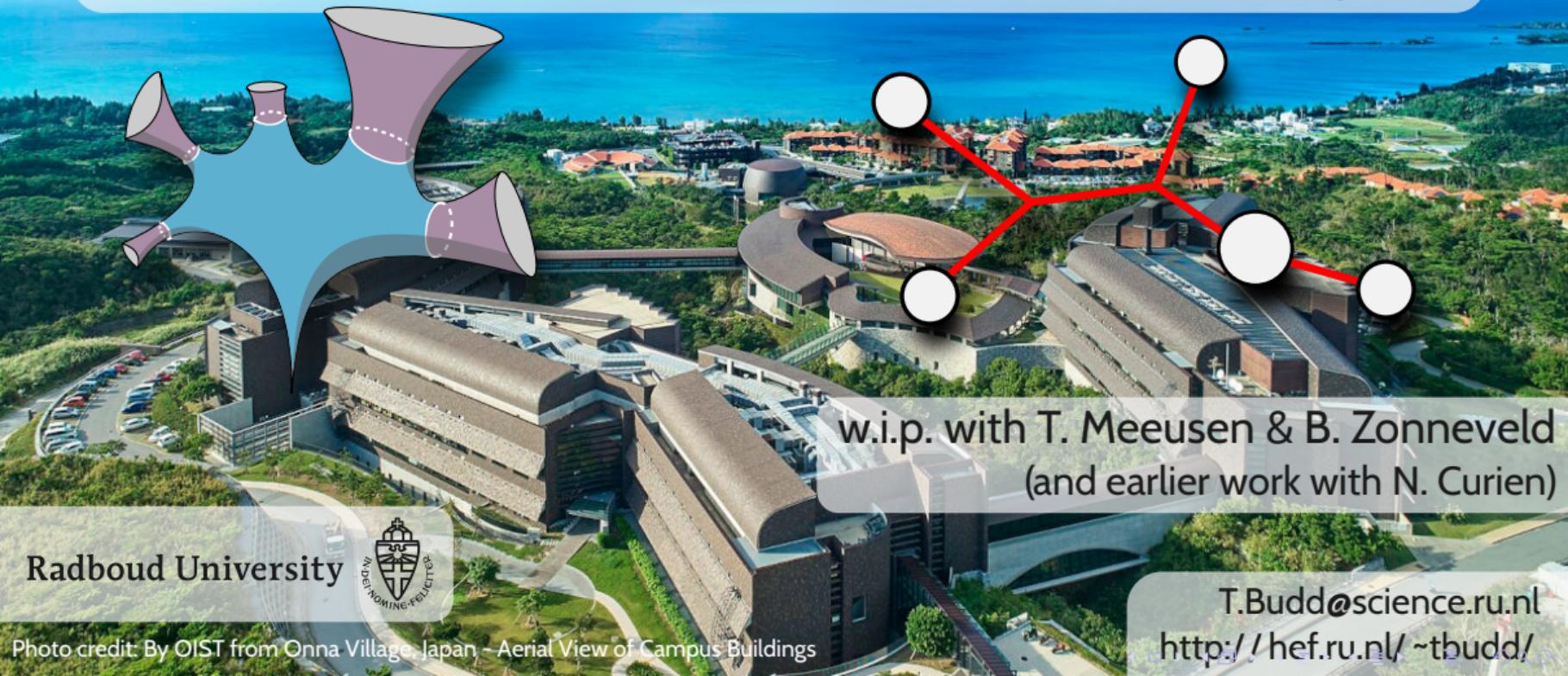


# Tree bijections and the geometry of random hyperbolic surfaces

Timothy Budd



Radboud University



Photo credit: By OIST from Onna Village, Japan - Aerial View of Campus Buildings

T.Budd@science.ru.nl

<http://hef.ru.nl/~tbudd/>

Putuo  
28°

Ningbo

Ninghai  
27°

Taizhou  
28°

Taipei  
27°

TAIWAN  
28°

Yonaguni  
29°

Miyakojima  
29°

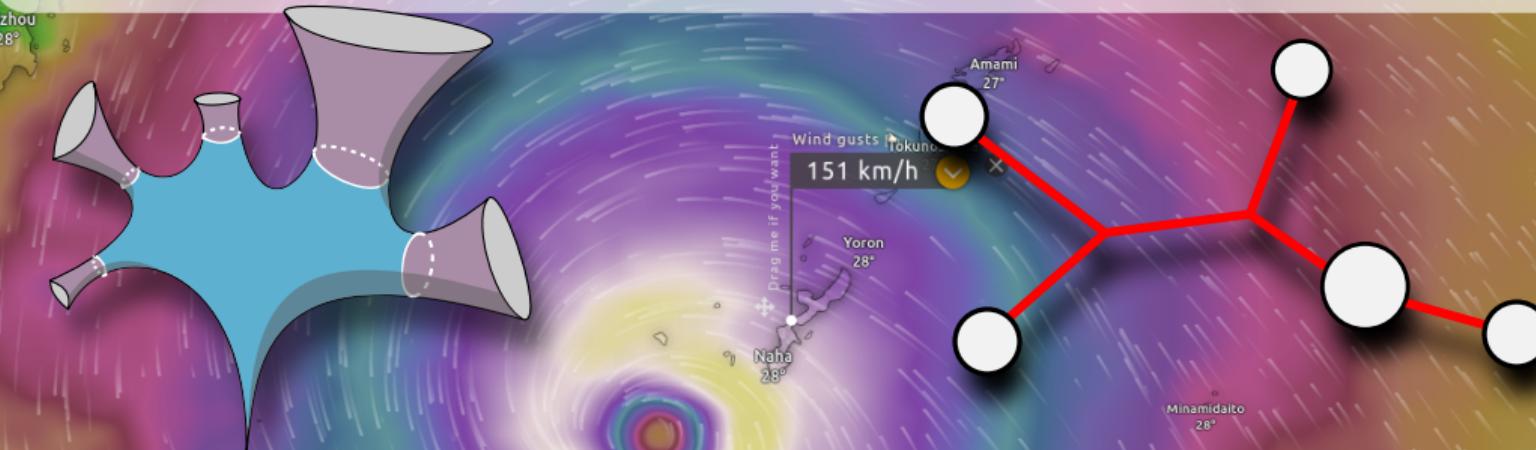


Radboud University

Image credit: windy.com

# Tree bijections and the geometry of random hyperbolic surfaces

Timothy Budd



w.i.p. with T. Meeusen & B. Zonneveld  
(and earlier work with N. Curien)

T.Budd@science.ru.nl

<http://hef.ru.nl/~tbudd/>

## Hyperbolic surfaces: a motivation from JT gravity

2D quantum gravity

$$Z = \int [Dg_{ab}] e^{-S[g]}$$



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JT gravity  Kazuhiro Sakai's talk!

[Teitelboim, '83] [Jackiw, '85]

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Constant curvature

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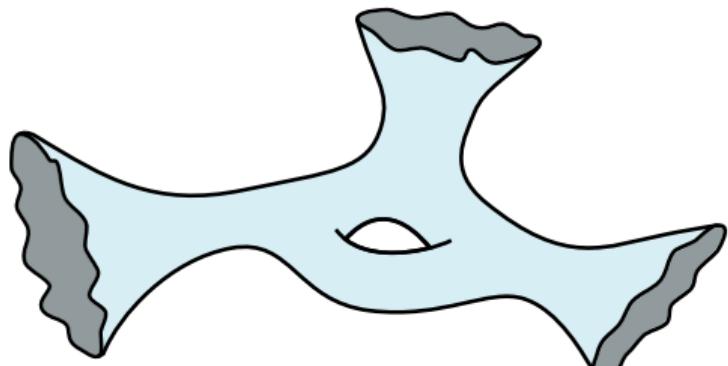
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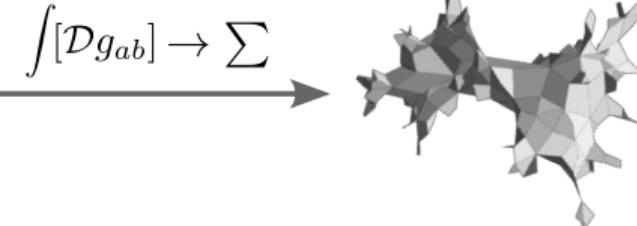
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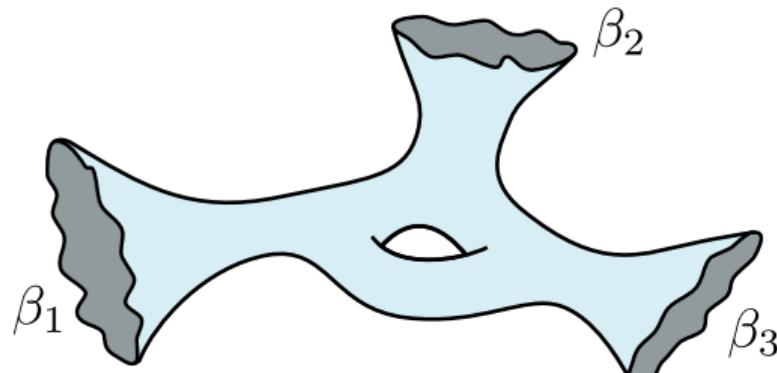
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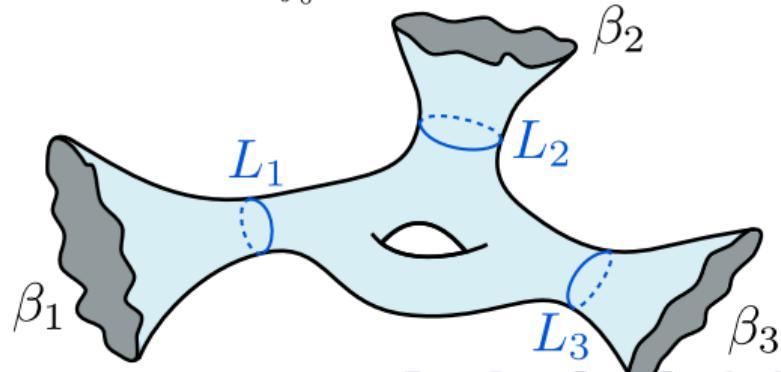
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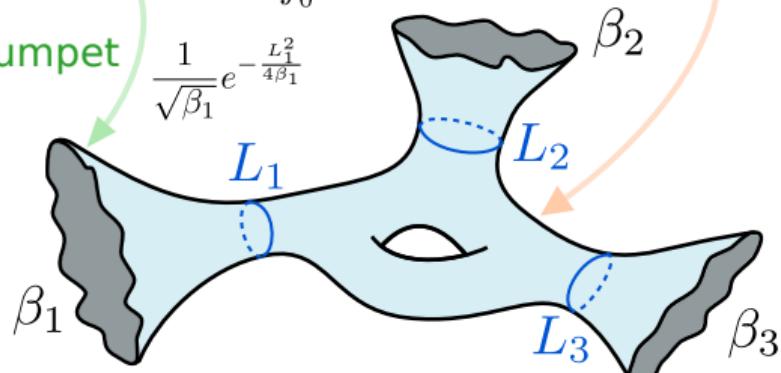


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trun

$$\frac{1}{\sqrt{\beta_1}} e^{-\frac{L_1^2}{4\beta_1}}$$

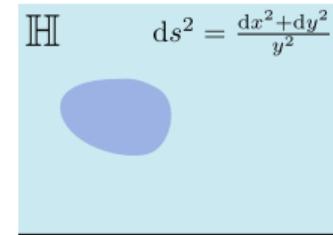
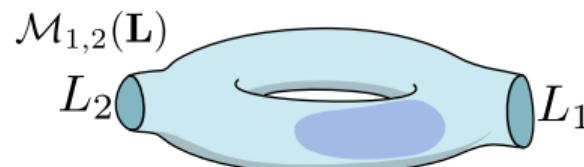
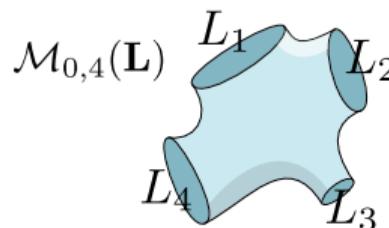


# The partition function of hyperbolic surfaces: WP volumes

[Wolpert, Penner, Zograf, Witten, Kontsevich, Mirzakhani, ...]

- ▶ Consider the **Moduli space**

$$\mathcal{M}_{g,n}(\mathbf{L}) = \left\{ \begin{array}{l} \text{hyperbolic metrics on genus-}g \text{ surface with } n \\ \text{geodesic boundaries of lengths } \mathbf{L} = (L_1, \dots, L_n) \end{array} \right\} / \text{Diff}^+$$



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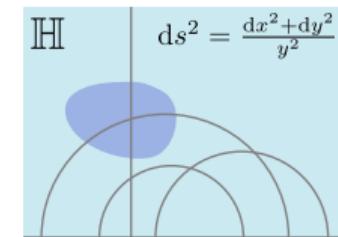
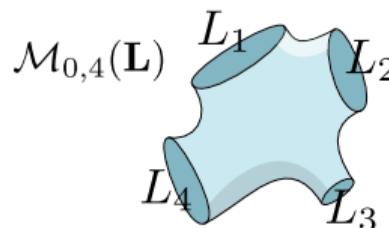
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Conformal equivalence classes  
of Riemannian metrics

also

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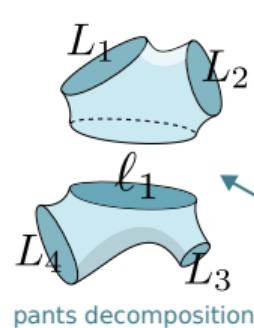
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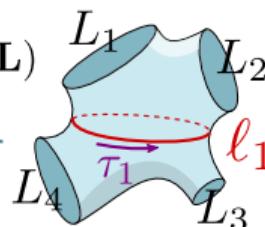
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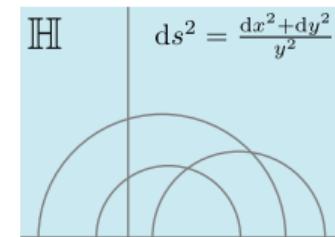
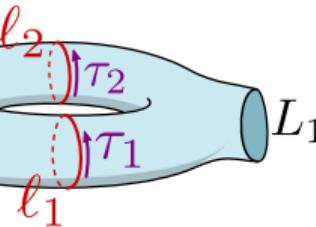
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$$\mathcal{M}_{1,2}(\mathbf{L})$$



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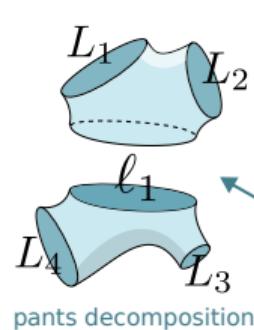
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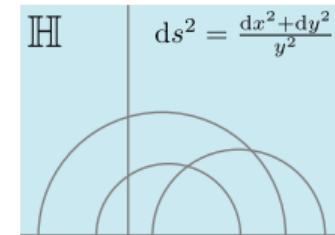
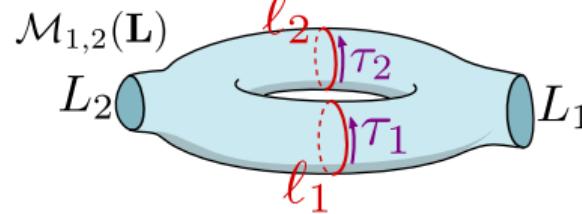
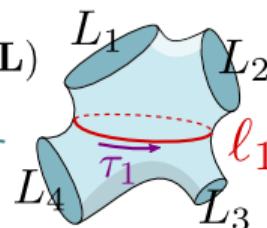
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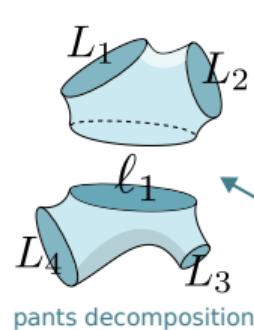
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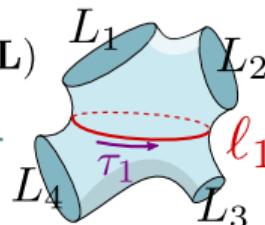
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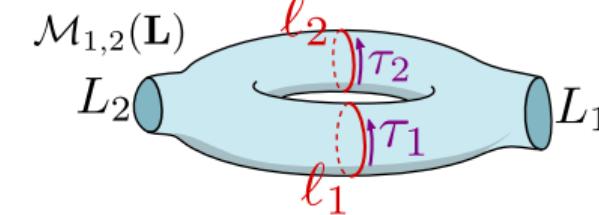
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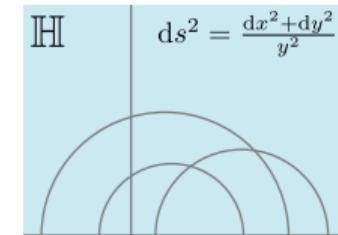
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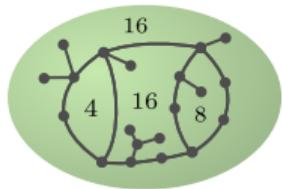
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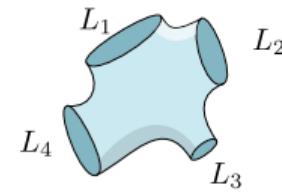
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- ▶ Examples:  $V_{0,3}(\mathbf{L}) = 1, \quad V_{0,4}(\mathbf{L}) = \frac{1}{2}(L_1^2 + L_2^2 + L_3^2 + L_4^2) + 2\pi^2,$   
 $V_{1,2}(\mathbf{L}) = \frac{1}{192}(L_1^2 + L_2^2 + 4\pi^2)(L_1^2 + L_2^2 + 12\pi^2).$

## Bipartite maps on surfaces



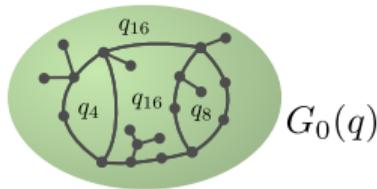
## Hyperbolic surfaces



## Bipartite maps on surfaces

- (grand canonical) partition function

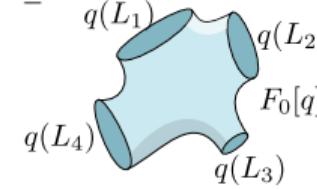
$$G_g(q) = \sum_{n \geq 1} \frac{1}{n!} \sum_{d_1=0}^{\infty} q^{2d_1} \cdots \sum_{d_n=0}^{\infty} q^{2d_n} \# \left\{ \begin{array}{l} \text{genus-}g \text{ maps with} \\ \text{face degrees } 2d_1, \dots, 2d_n \end{array} \right\}$$



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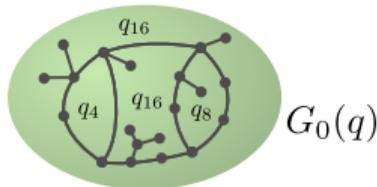
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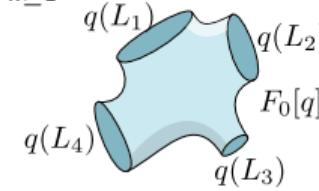
$e^{\sum_g \lambda^g G_g}$  is a  $\tau$ -function of KP hierarchy

[Kadomtsev, Petriashvili, Panharipande, Okounkov, Kazarian, ...]

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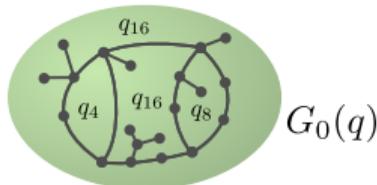
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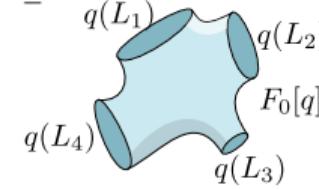
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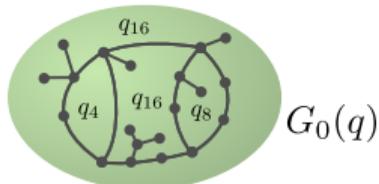
$e^{\sum_g \lambda^g F_g}$  is a  $\tau$ -function of KdV hierarchy

[Witten, '91][Kontsevich, '92][Kaufmann, Manin, Zagier, '96][Mirzakhani, '07]

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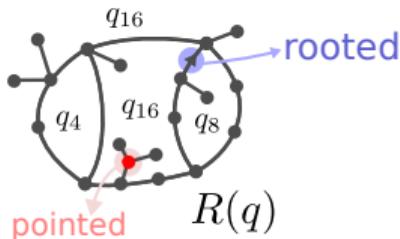
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$G_0(q)$

- $G_0$  determined by string eq. for  $R(q) := \frac{\partial G_0}{\partial q_0 \partial q_1}$

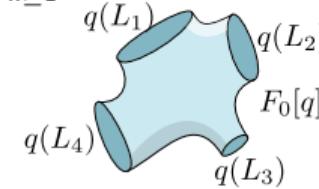
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$q(L_3)$

$q(L_4)$

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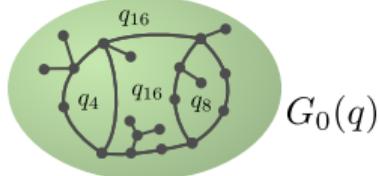
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&lt;p

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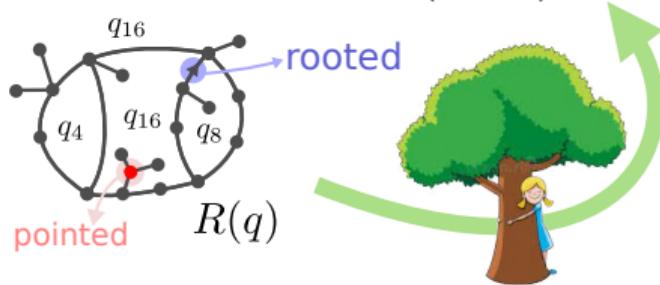
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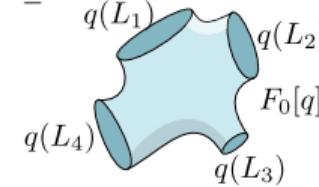
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$F_0[q]$

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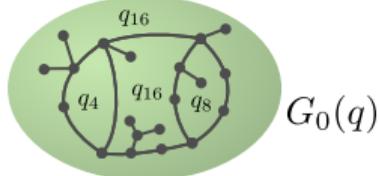
$q(L_4)$

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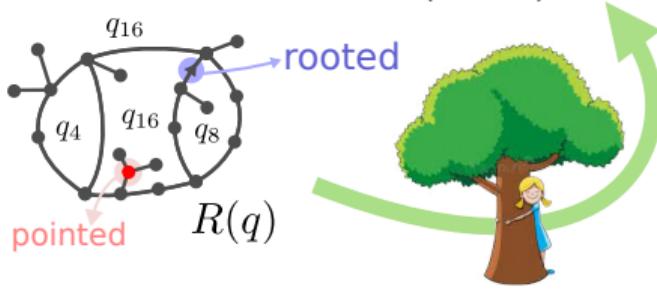
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- $G_0$  determined by string eq. for  $R(q) := \frac{\partial G_0}{\partial q_0 \partial q_1}$

$$R = 1 + \sum_{k=1}^{\infty} \binom{2k-1}{k} q^{2k} R^k$$



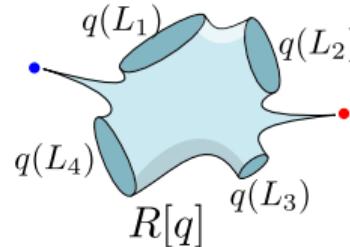
## Hyperbolic surfaces

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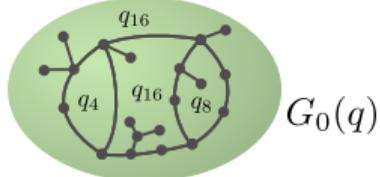
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## Bipartite maps on surfaces

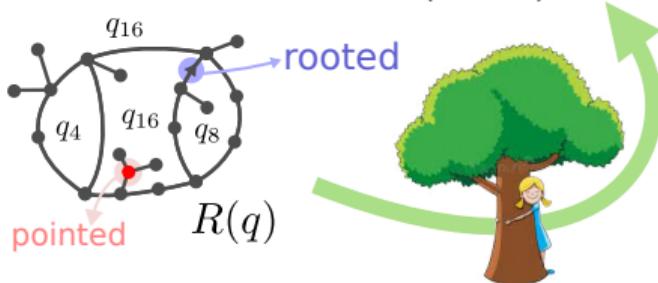
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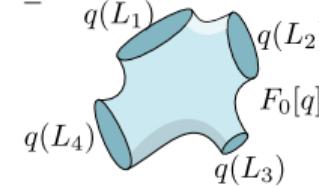
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## Hyperbolic surfaces

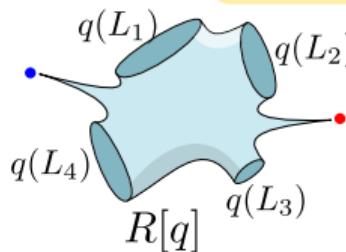
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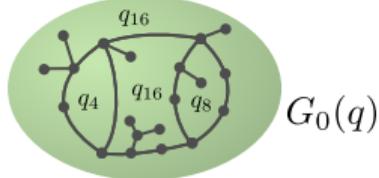
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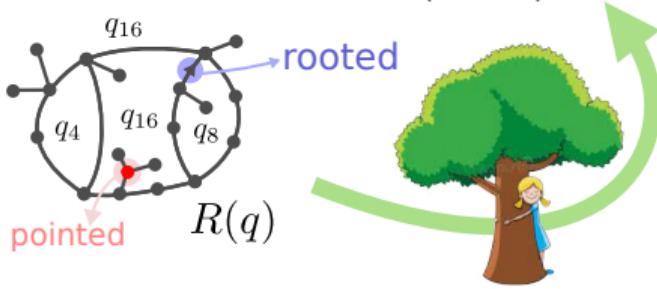
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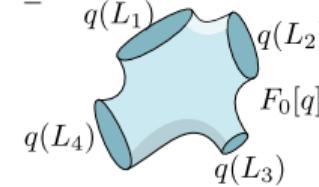
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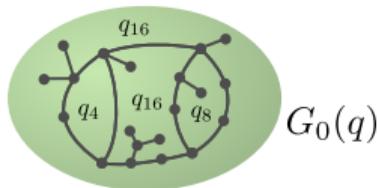
$\frac{2}{k!} \int_0^{\infty} \left(\frac{L}{2}\right)^{2k} dq(L)$ 
 $\frac{(-1)^k \pi^{2k-2}}{(k-1)!} \mathbf{1}_{k \geq 2}$

$R[q]$

## Bipartite maps on surfaces

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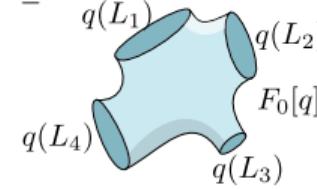
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- $G_0 \xrightarrow{\text{probability}}$  Boltzmann planar map  $\mathfrak{m}$

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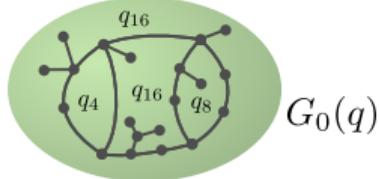


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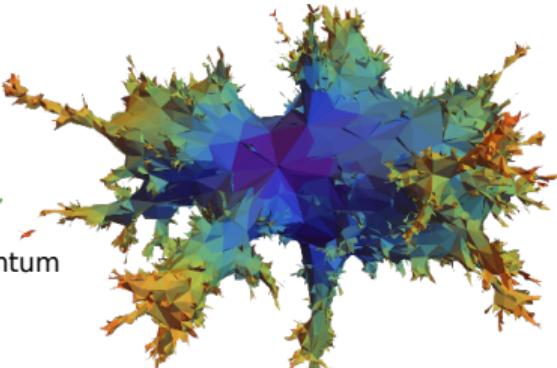
► Random metric space

► Hausdorff dimension 4

► Topology of 2-sphere

[Le Gall, Miermont, Marckert, Marzouk, ...]

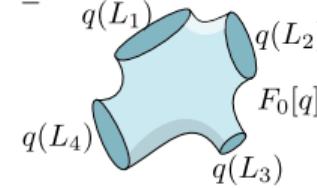
► Metric of Liouville Quantum Gravity at  $\gamma = \sqrt{8/3}$   
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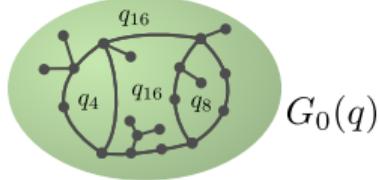


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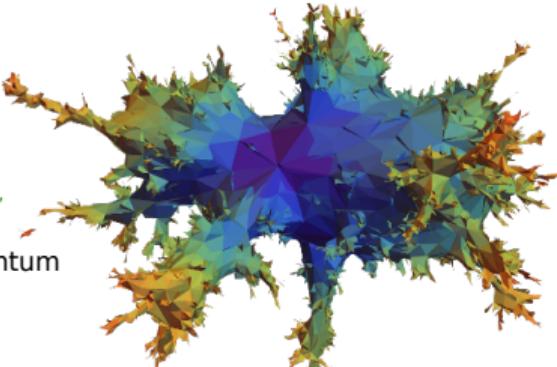
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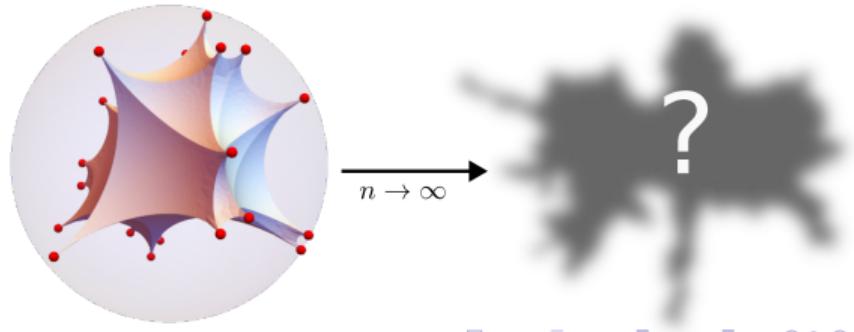


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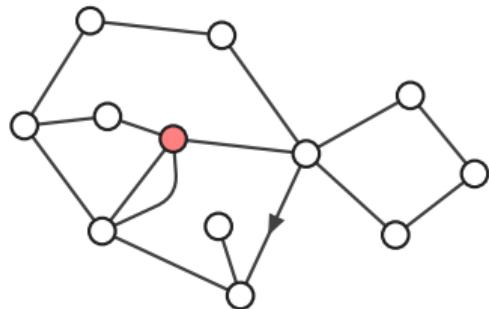
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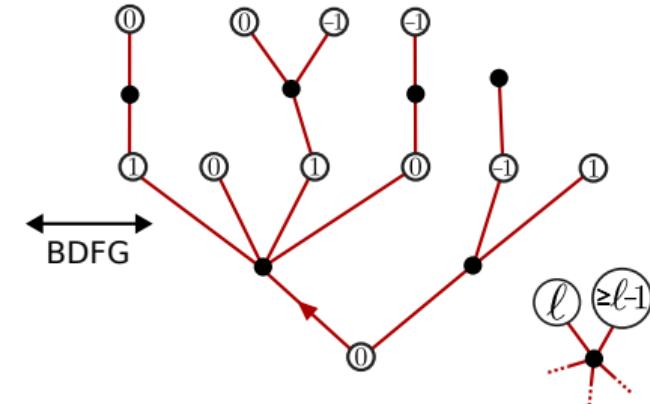


## Bouttier–Di Francesco–Guitter bijection [BDFG, '04]

$$\left\{ \begin{array}{l} \text{rooted bipartite planar maps} \\ \text{with marked vertex ("origin")} \end{array} \right\} \xleftrightarrow{2\text{-to-1}} \left\{ \begin{array}{l} \text{mobiles (bicolored plane trees} \\ \text{with labeled white vertices)} \end{array} \right\}$$

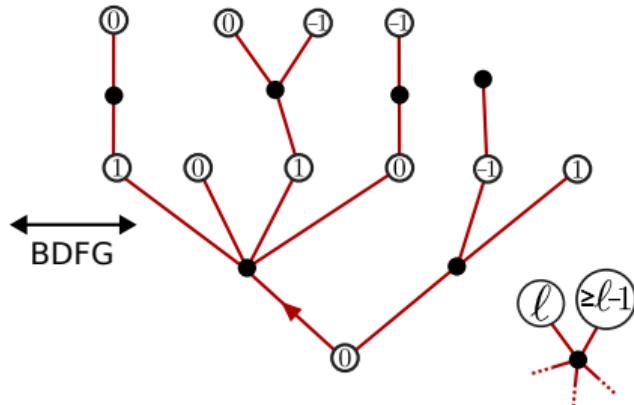
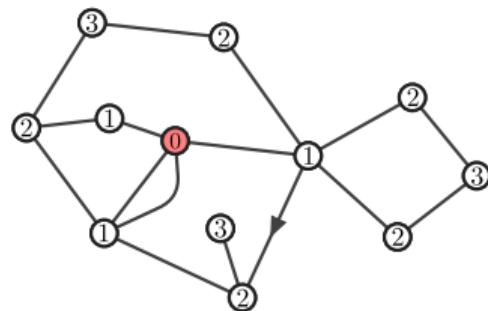


BDFC



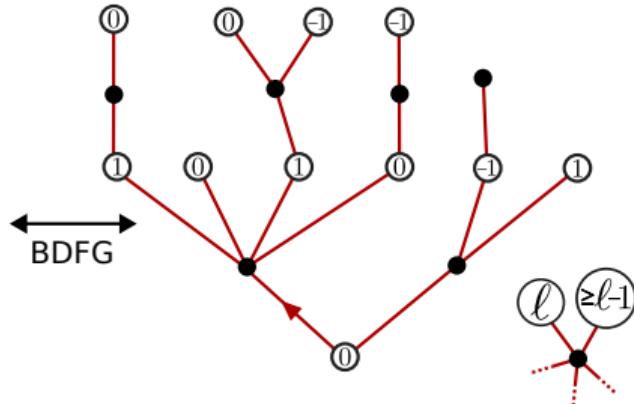
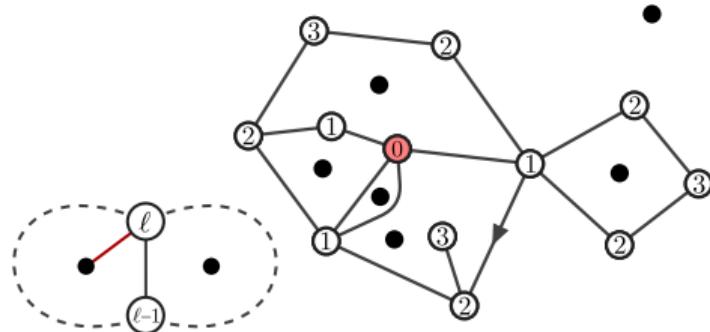
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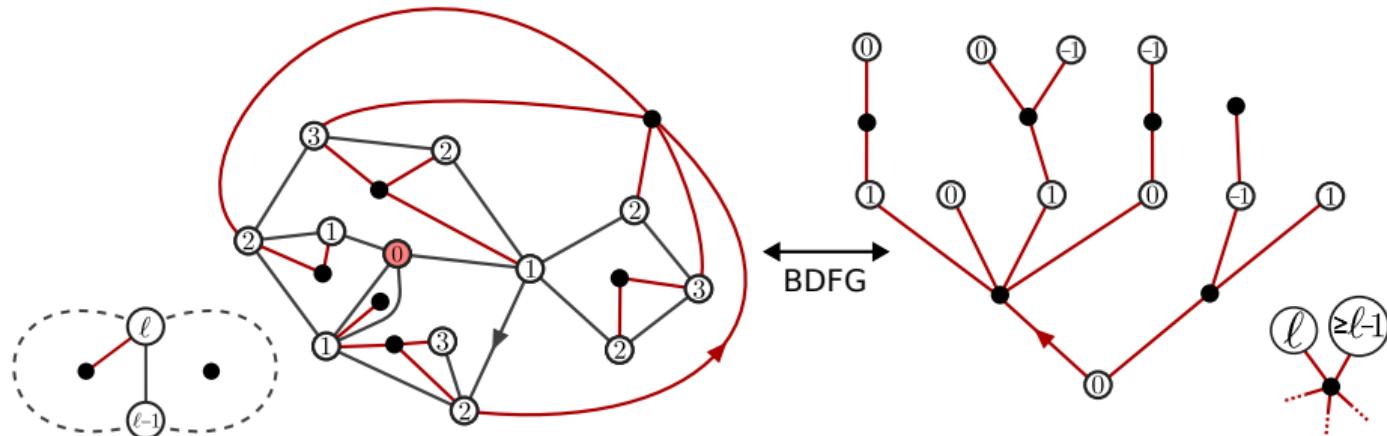
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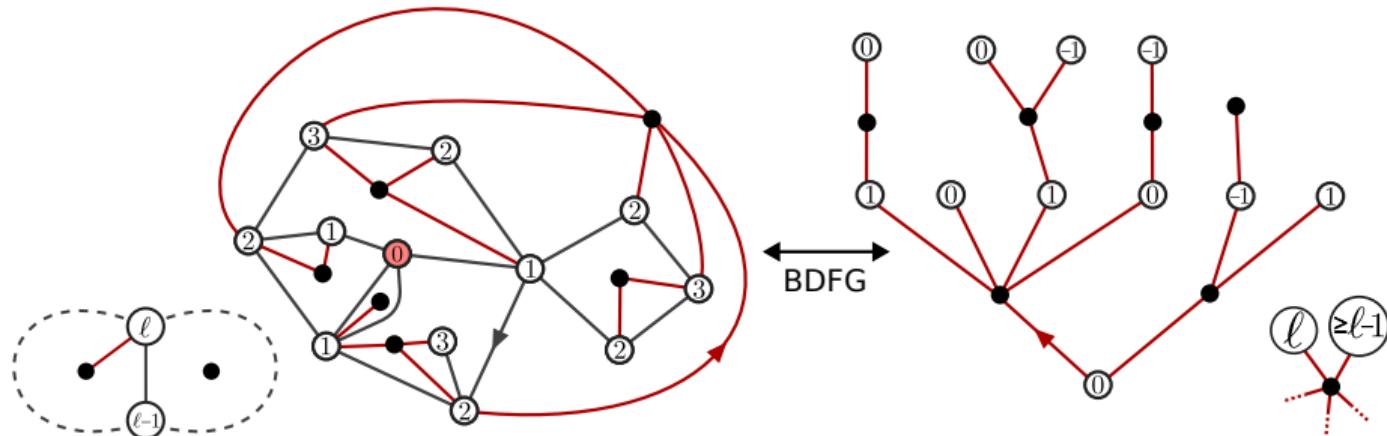
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► Face of degree  $2k$     $\longleftrightarrow$    Black vertex of degree  $k$ .

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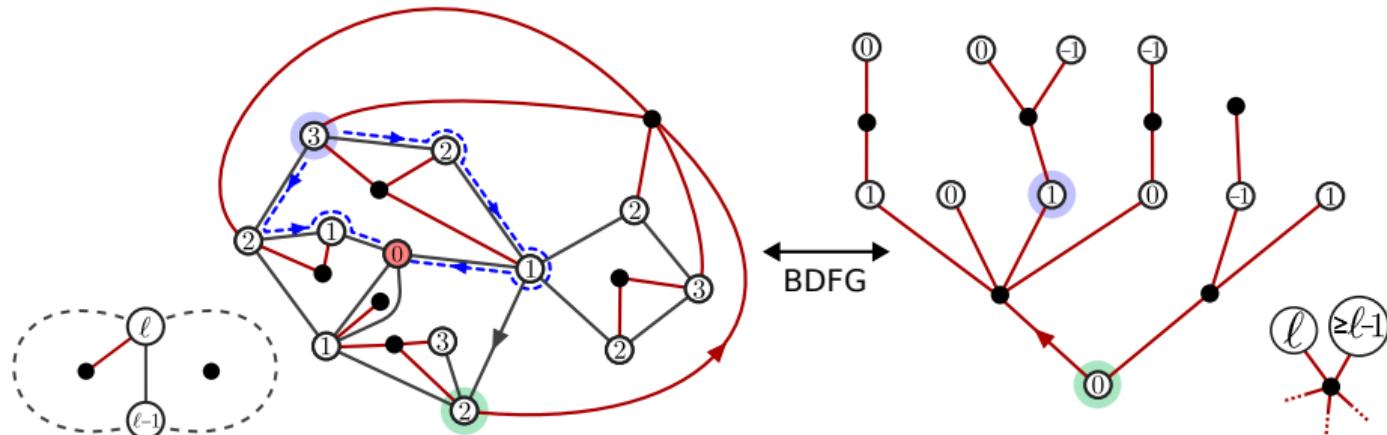


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$$R = @ + \sum_{k=1}^{\infty} q_{2k} \sum_{\text{labels}} \text{ (Diagram showing a black vertex of degree } k \text{ connected to } k \text{ labels, each labeled } R_{\text{@}}) = 1 + \sum_{k=1}^{\infty} q_{2k} \binom{2k-1}{k} R^k,$$

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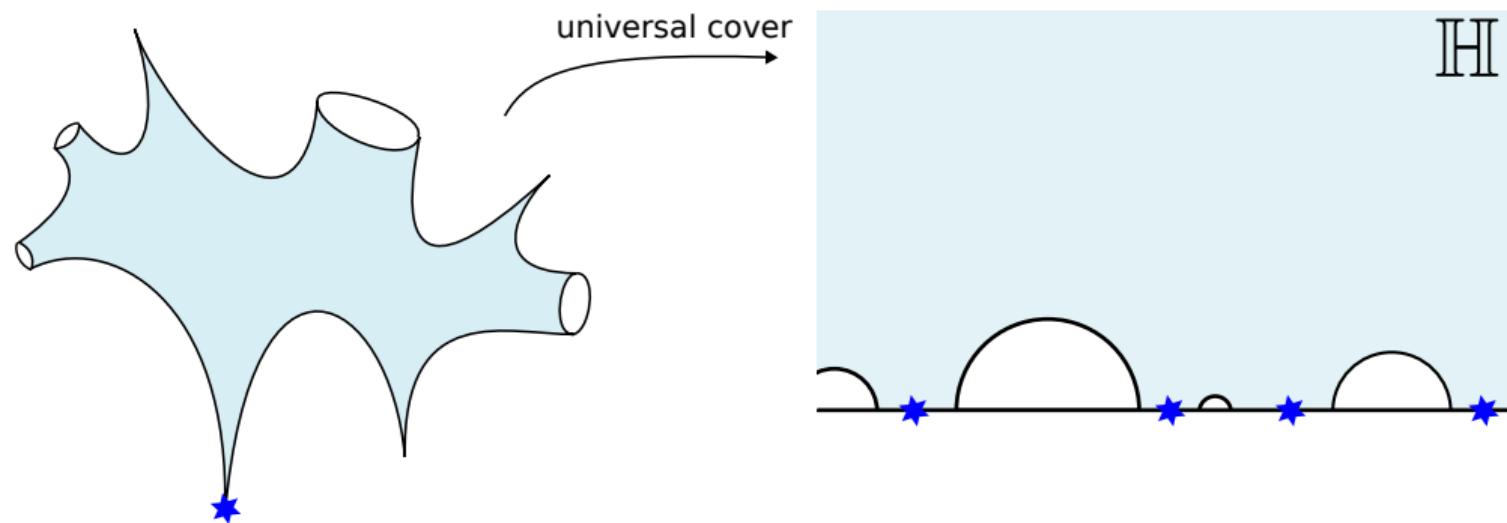


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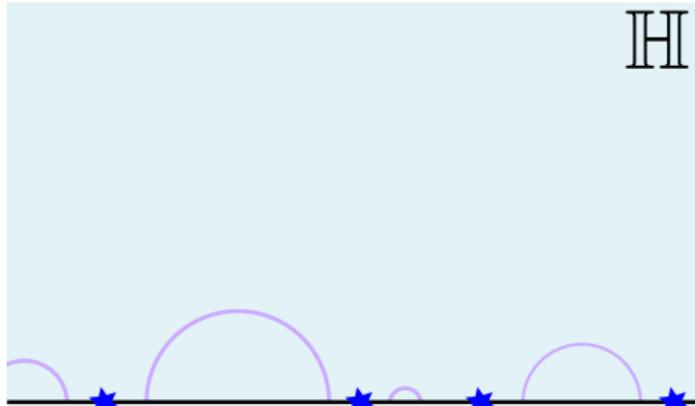
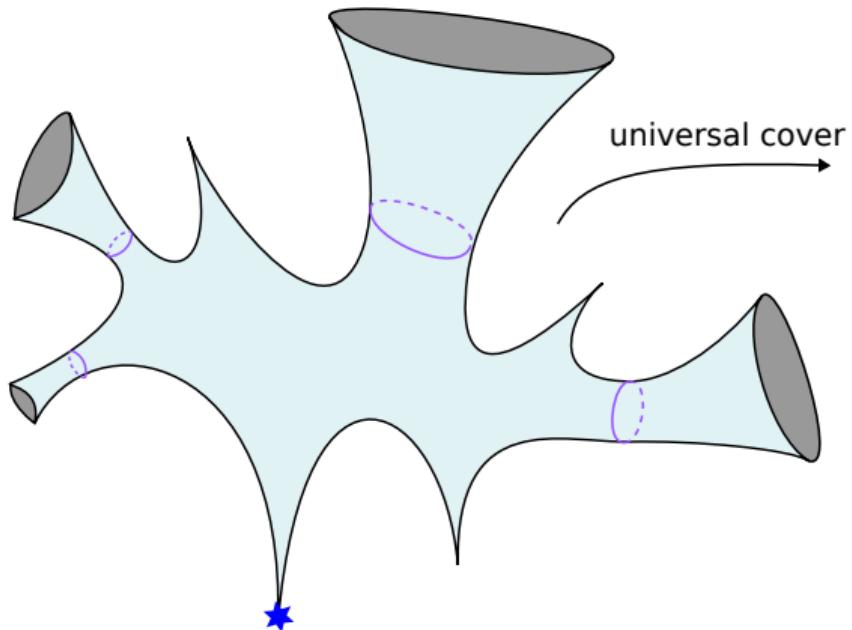
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► Vertex at distance  $r > 0$  to origin  $\longleftrightarrow$  White vertex with label  $r - r_{\text{root}}$ .

## Tree in a hyperbolic surface?

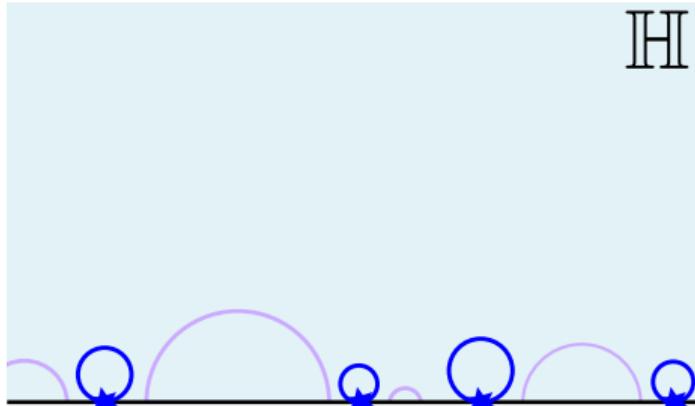
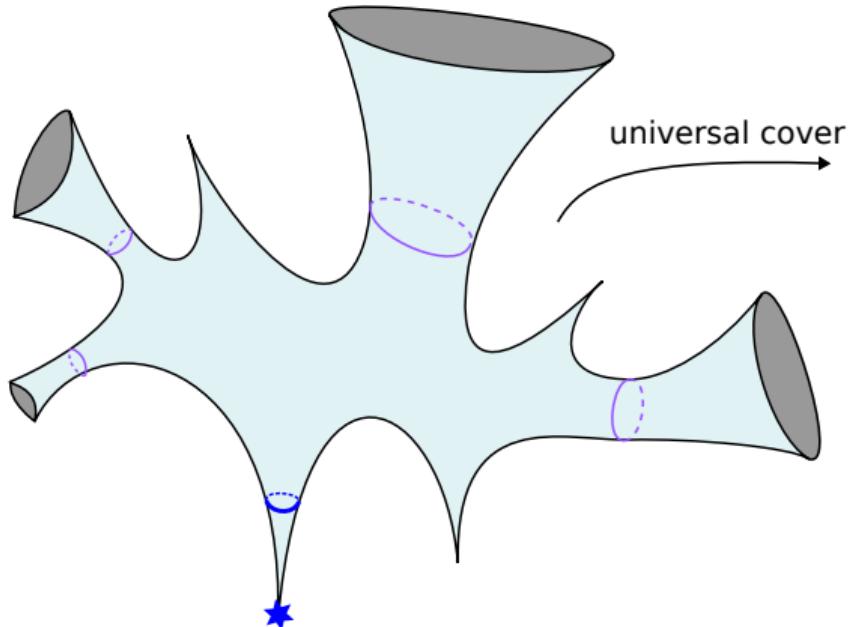


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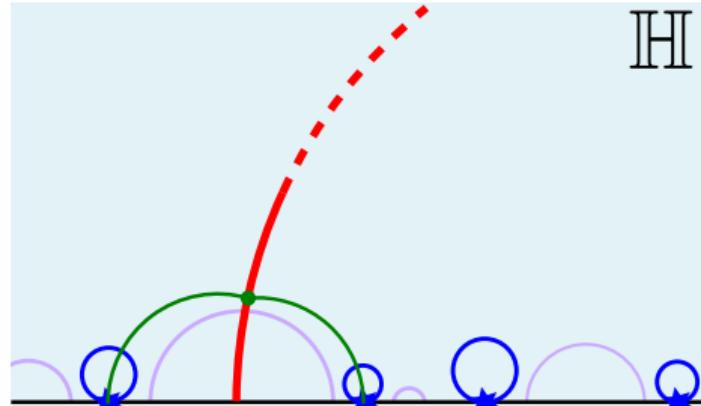
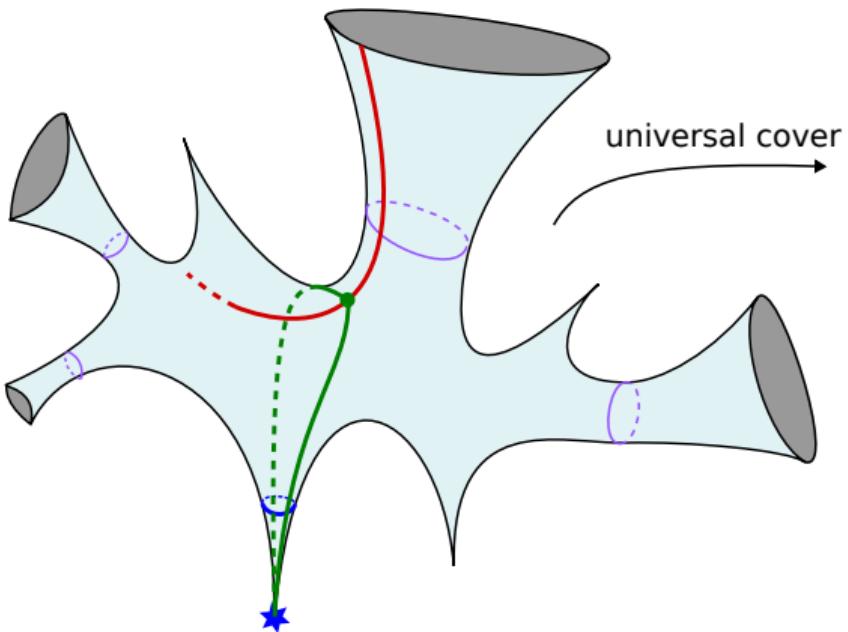
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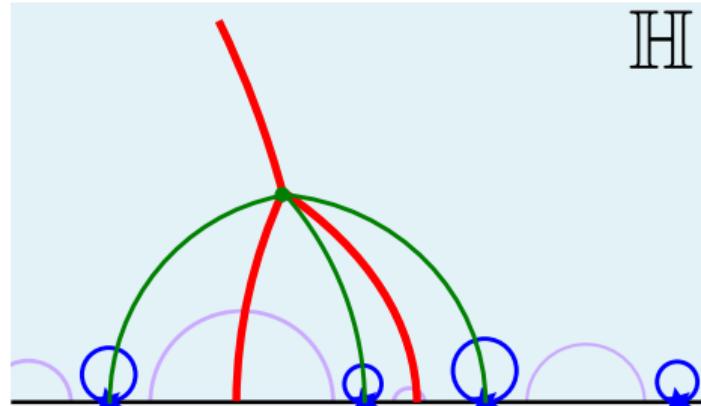
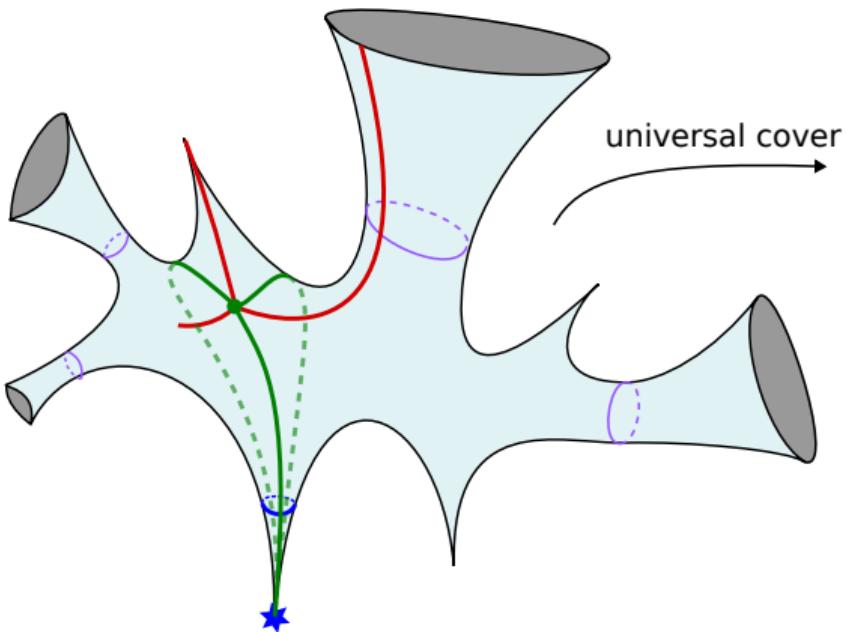
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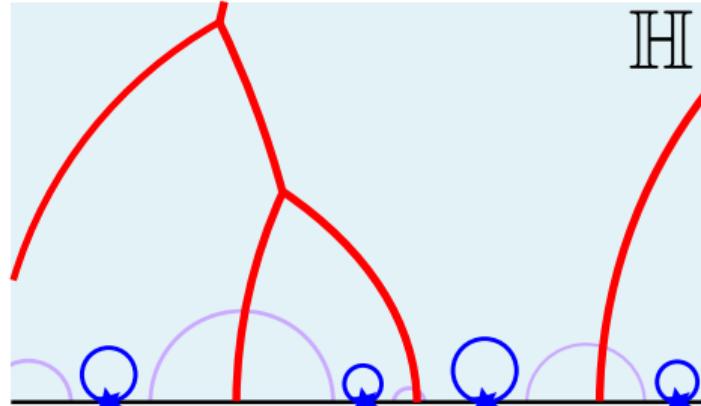
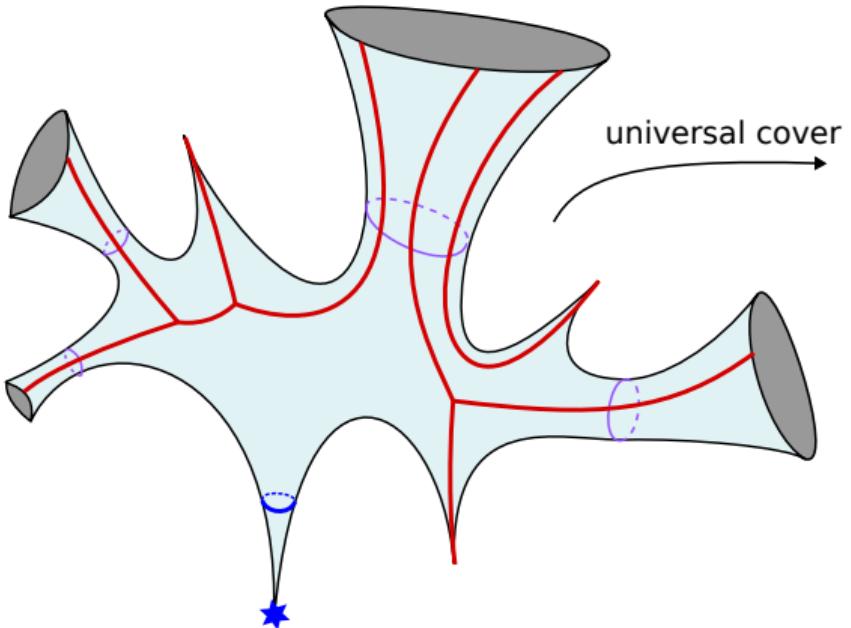
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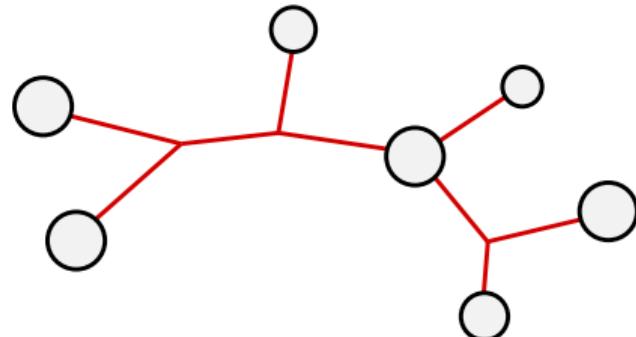
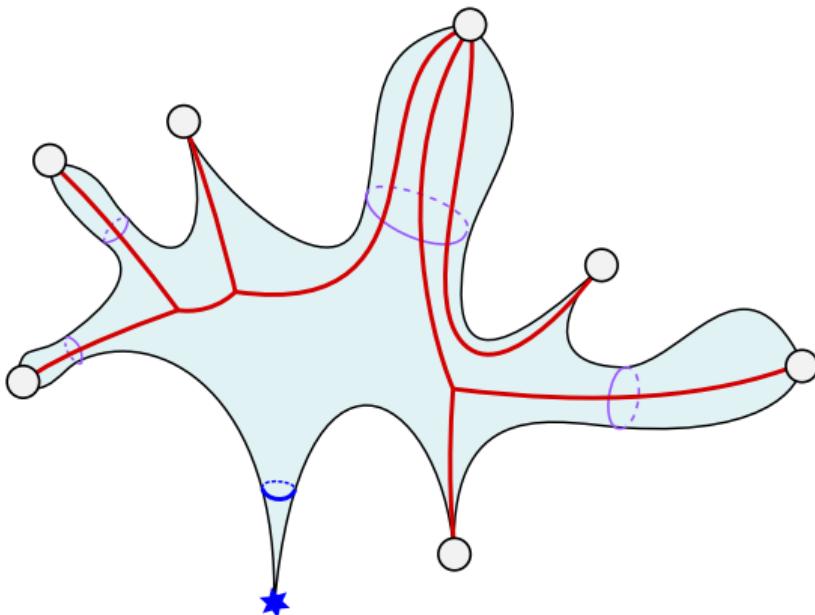
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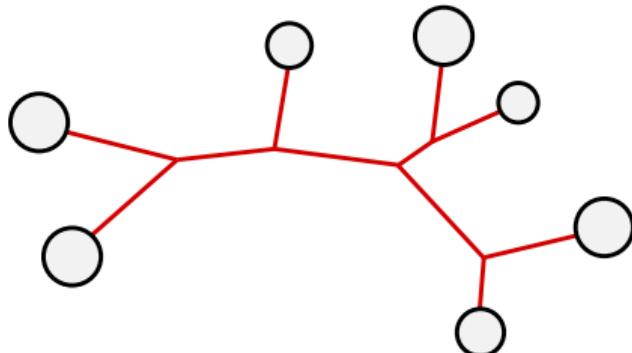
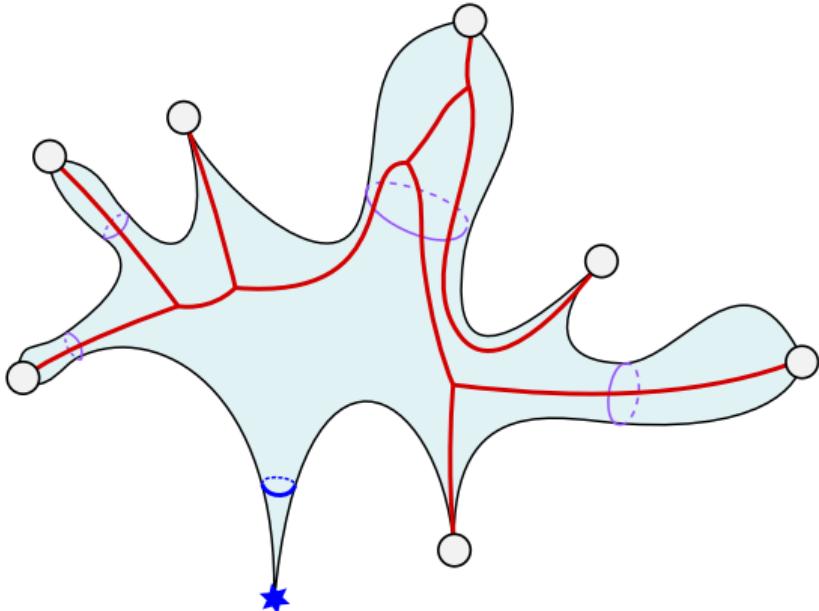
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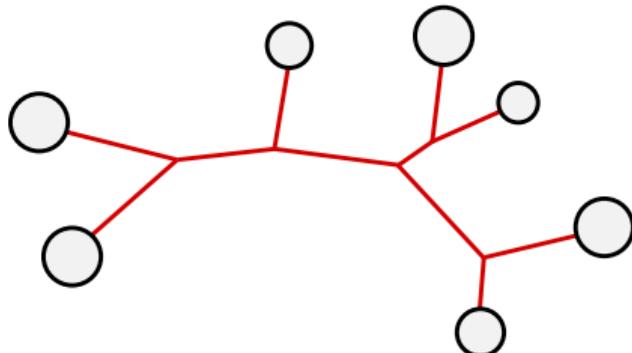
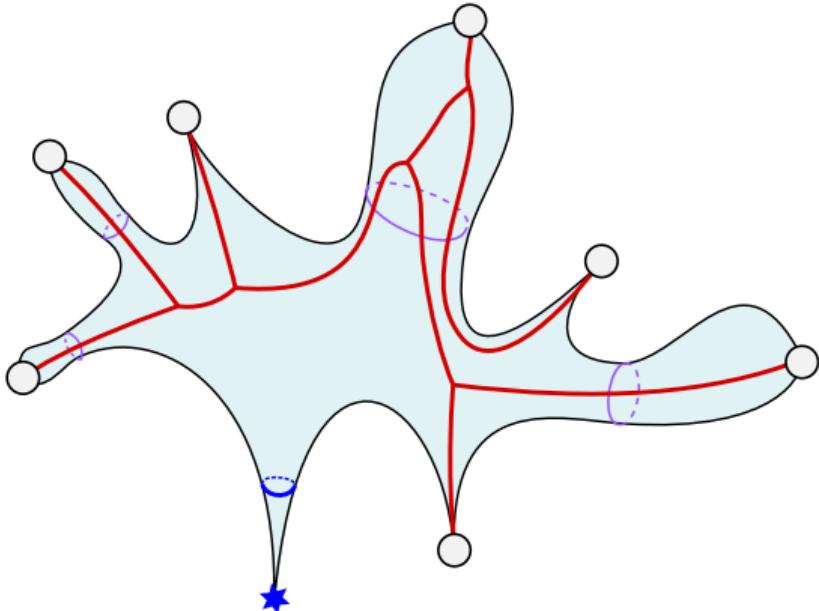
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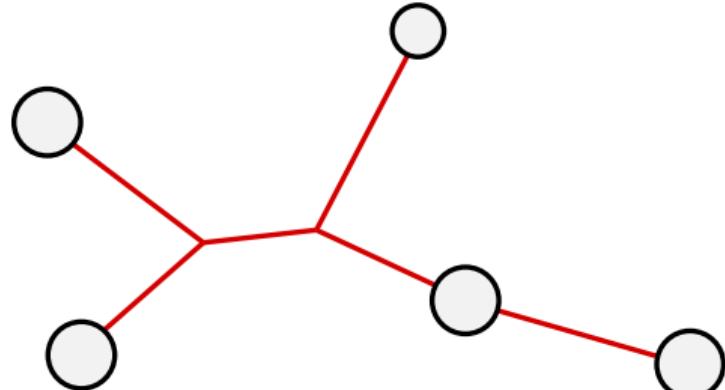
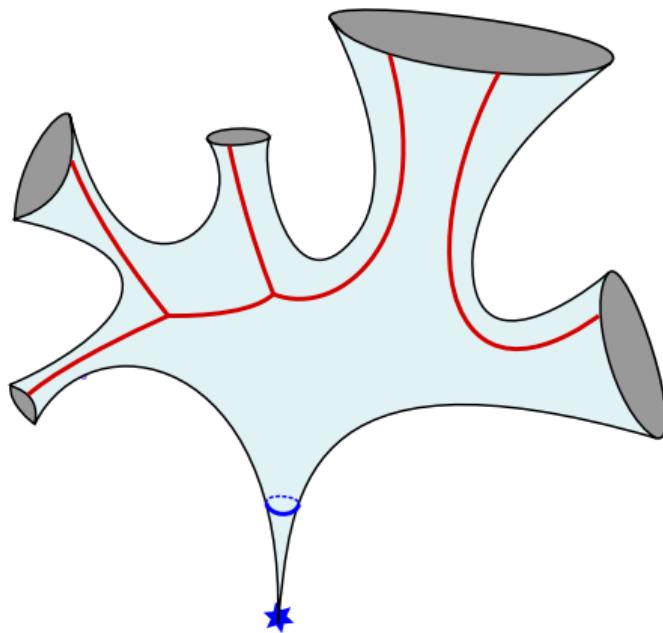
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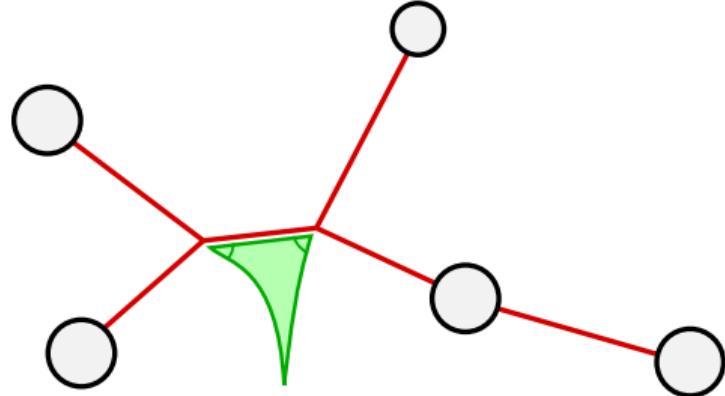
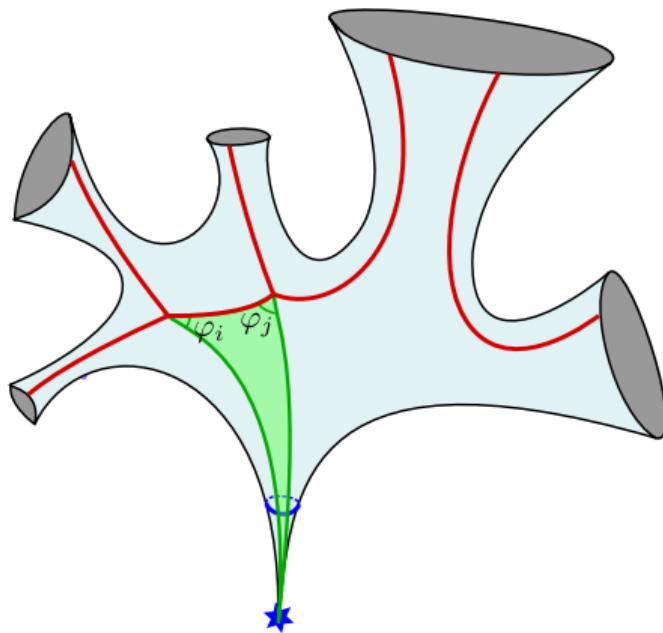
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- ▶ Can we label the tree to make a bijection?

Labels: angles on half edges



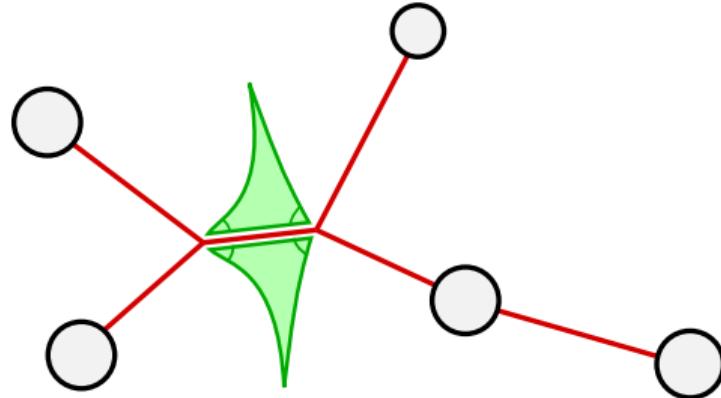
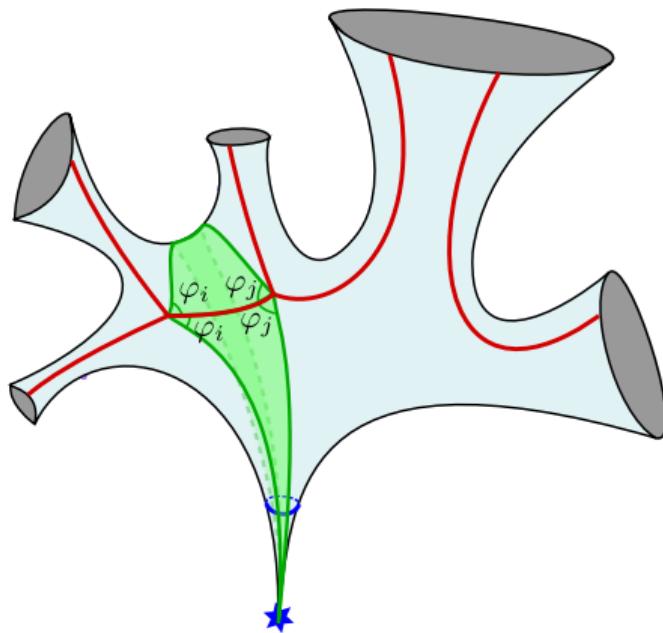
- ▶ The surface is canonically triangulated by

## Labels: angles on half edges



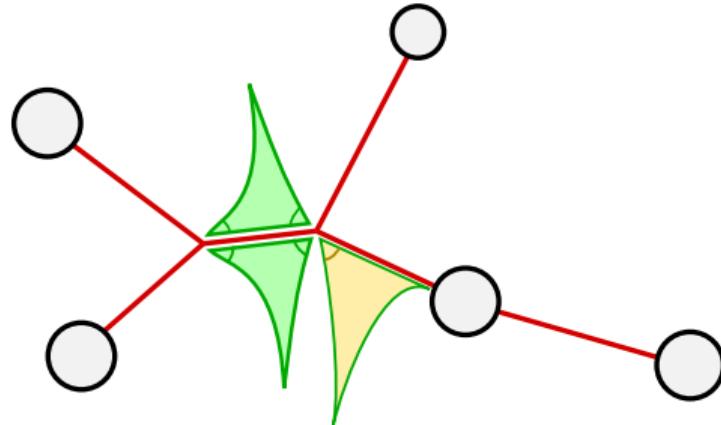
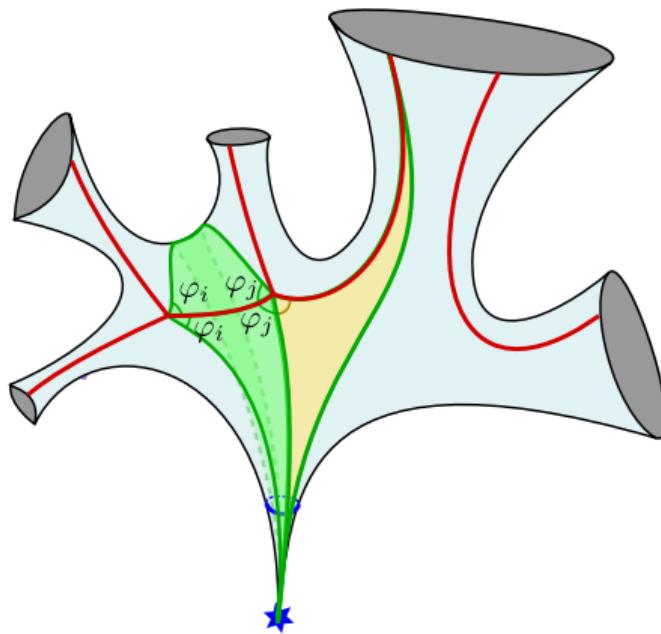
- ▶ The surface is canonically triangulated by
  - ▶ for each spine edge: triangle with angles  $\varphi_i, \varphi_j, 0$  (so  $\varphi_i + \varphi_j < \pi$ )

Labels: angles on half edges



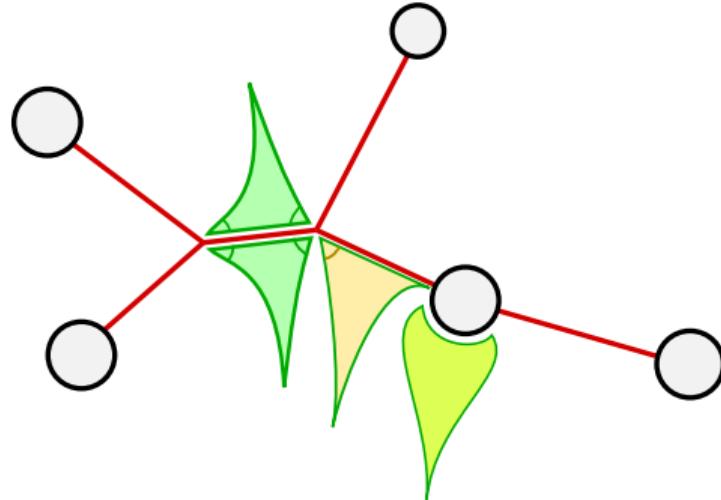
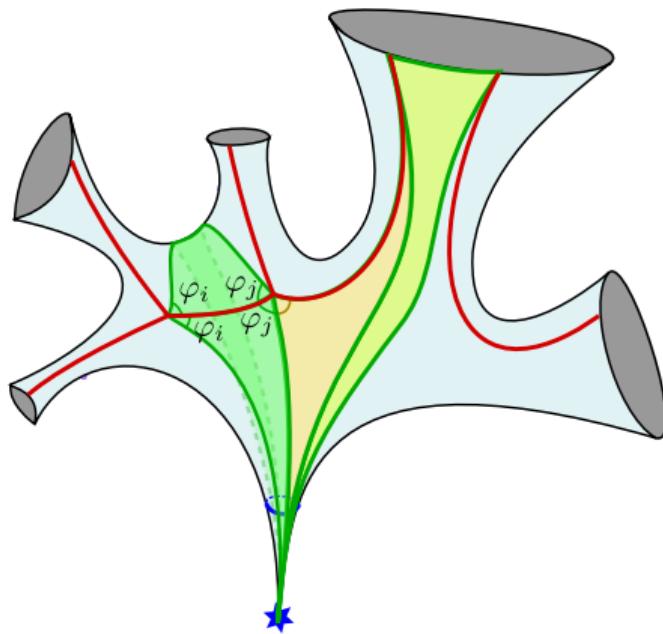
- ▶ The surface is canonically triangulated by
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## Labels: angles on half edges



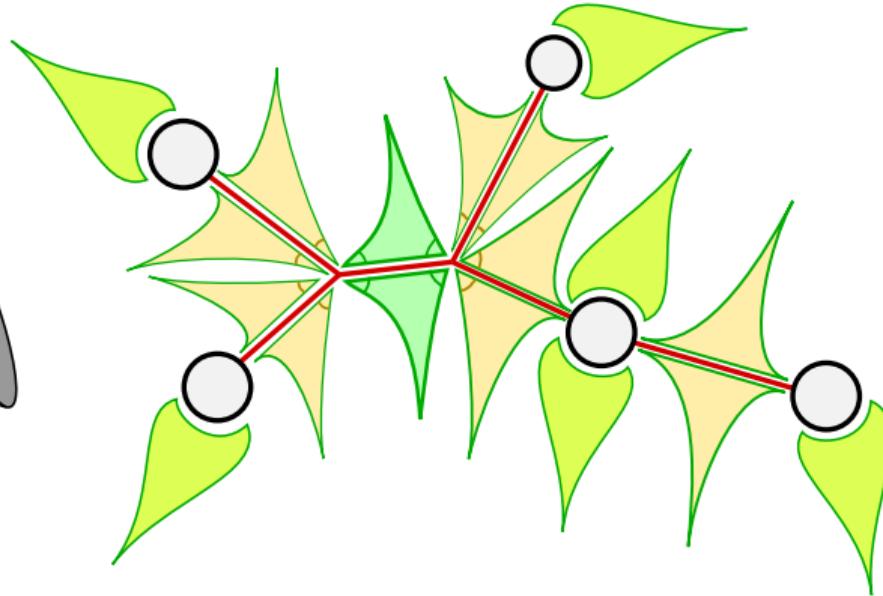
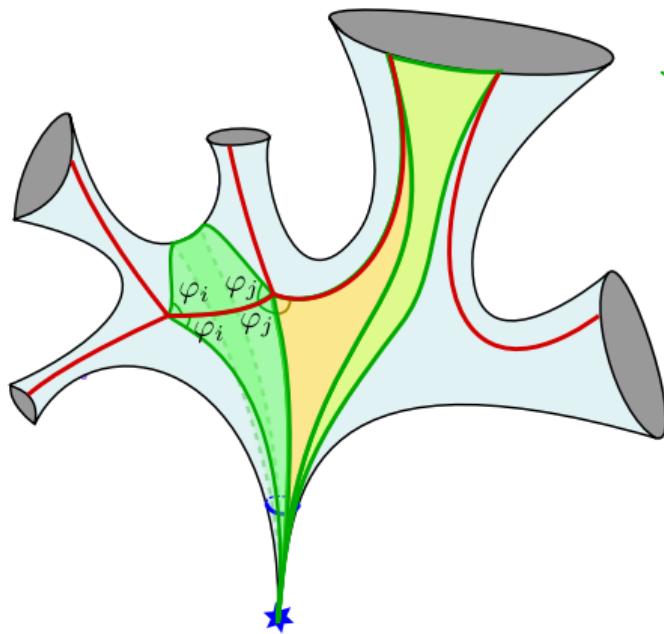
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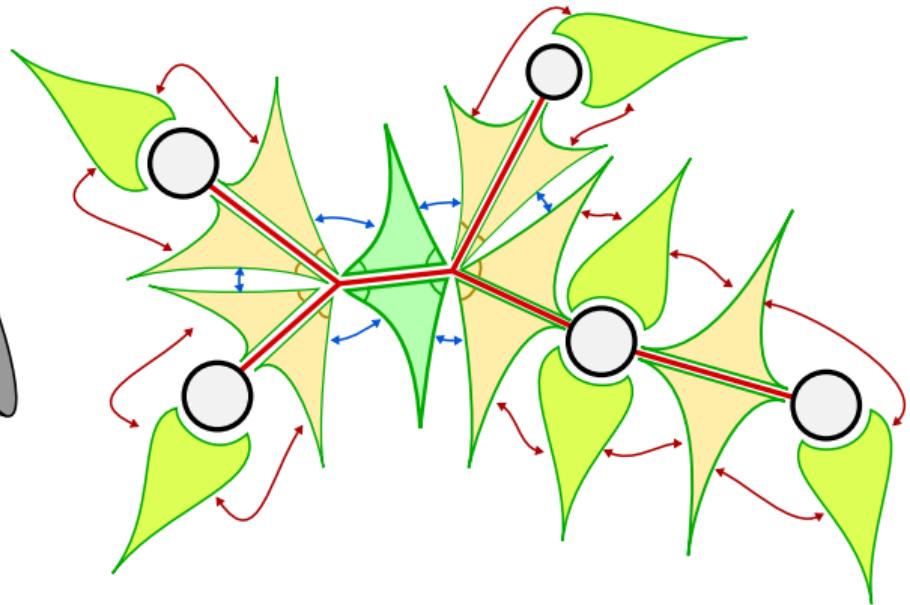
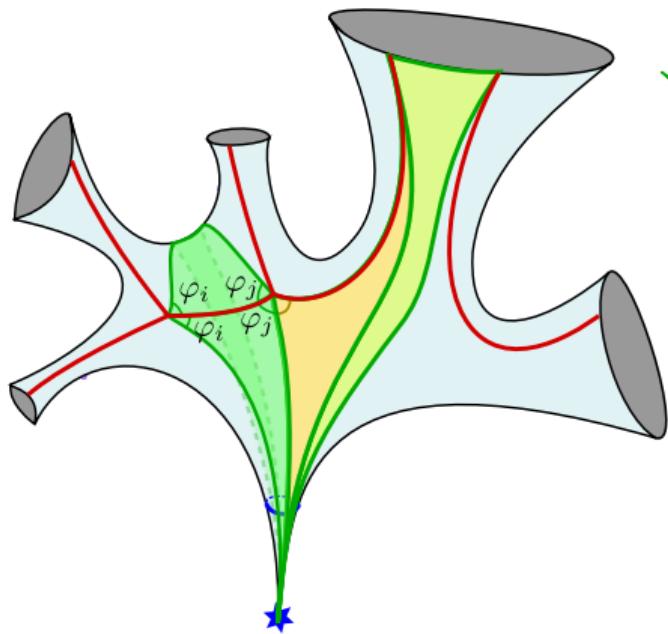
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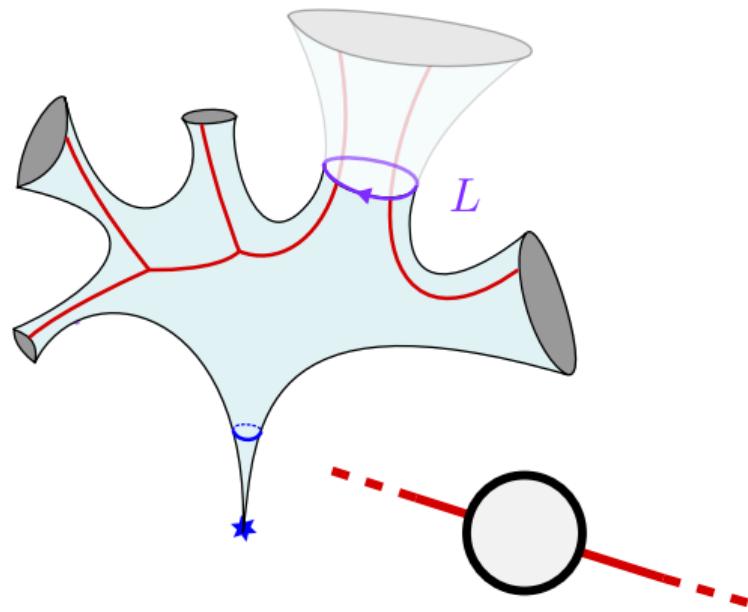
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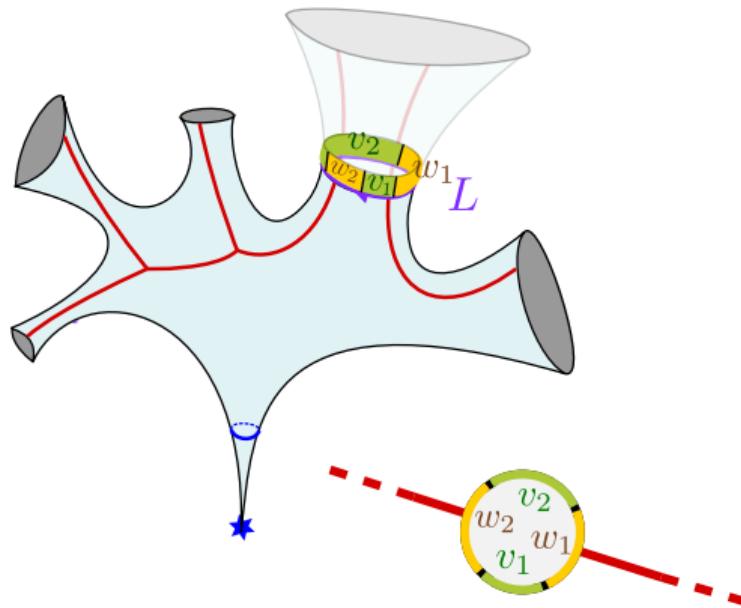


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  - ▶ for each corner of white vertex: an ideal wedge.
- ▶ Gluing of triangles is unique, except for **bi-infinite sides**: need extra parameters for injectivity.

## Labels: geometry around boundary

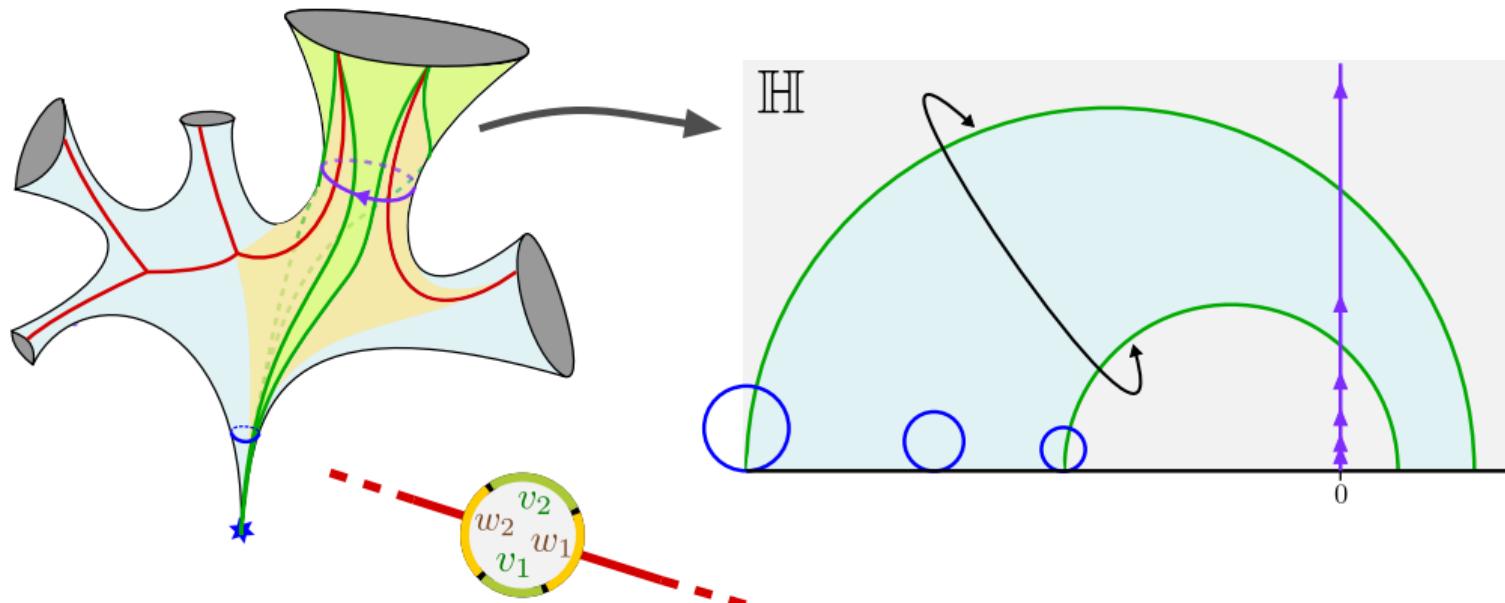


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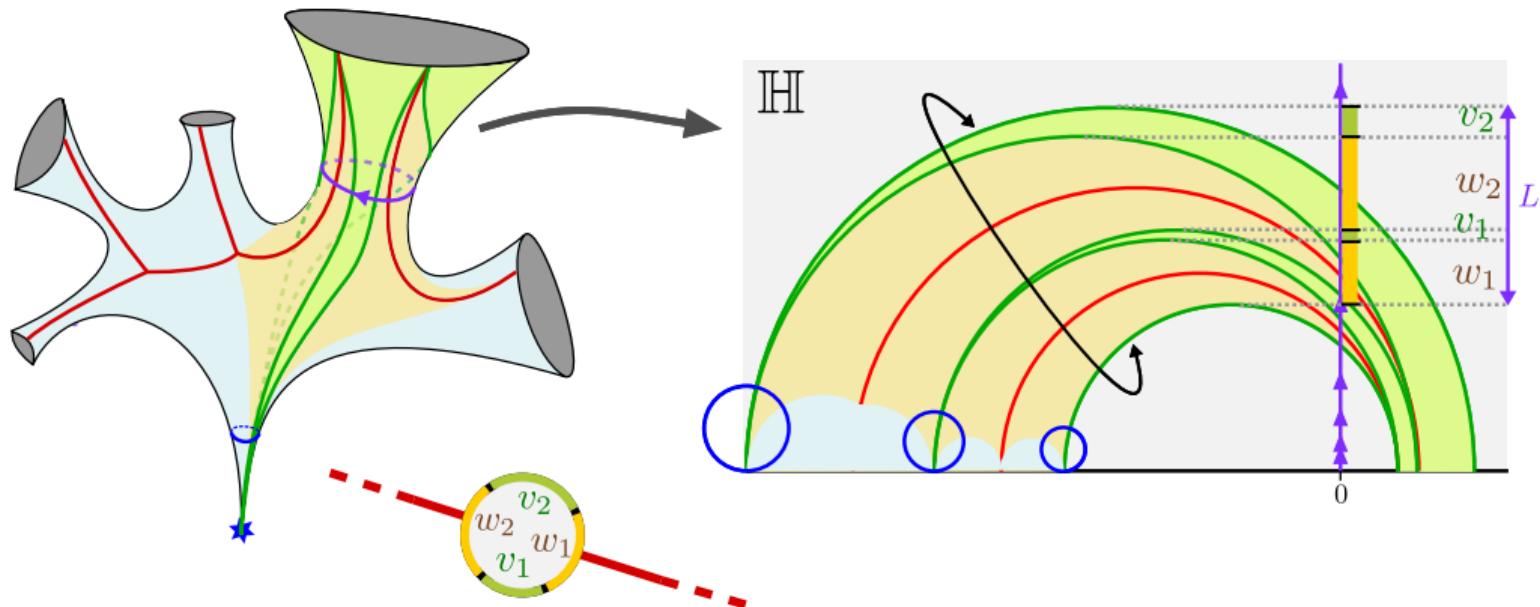
- Boundary of degree  $k$  partitions into  $2k$  segments of lengths  $v_1, \dots, v_k, w_1, \dots, w_k$ .

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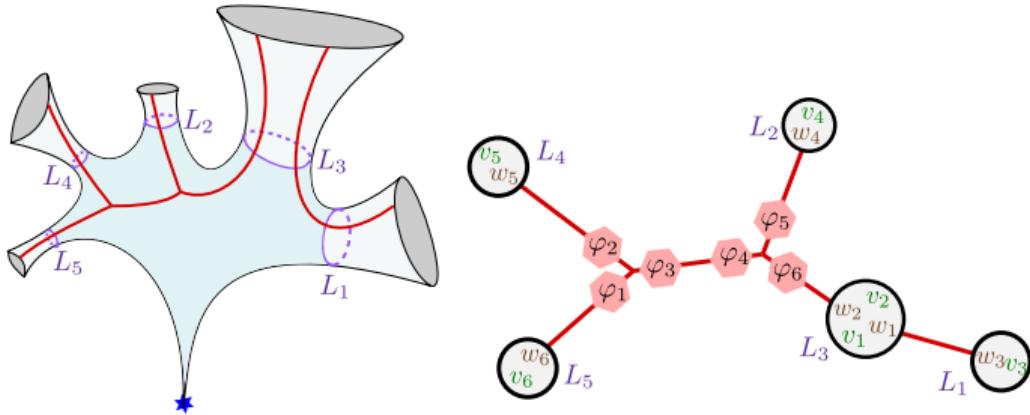
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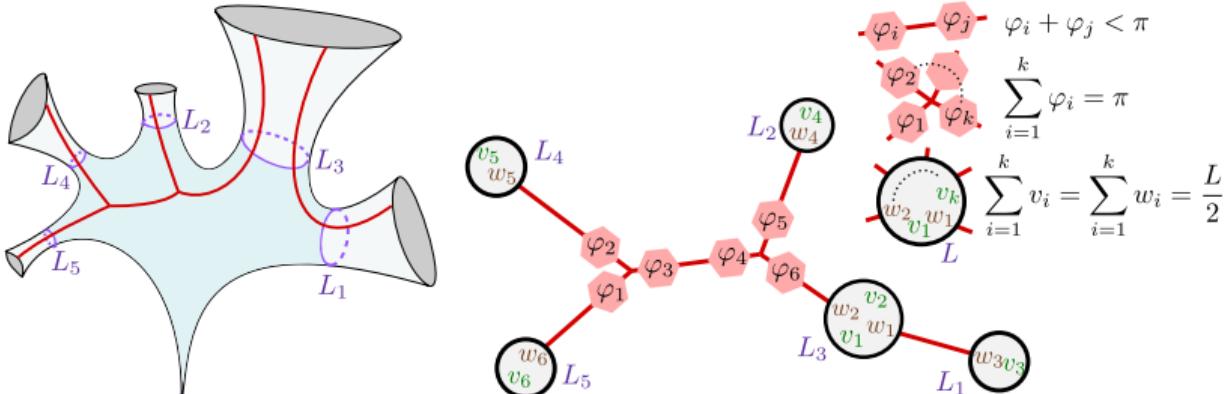
- ▶ Boundary of degree  $k$  partitions into  $2k$  segments of lengths  $v_1, \dots, v_k, w_1, \dots, w_k$ .
- ▶ Uniquely determines gluing, so should label vertex by

$$\left\{ (\mathbf{v}_i, \mathbf{w}_i)_{i=1}^k : \sum_{i=1}^k v_i = \sum_{i=1}^k w_i = \frac{L}{2} \right\}.$$

## Bijection result



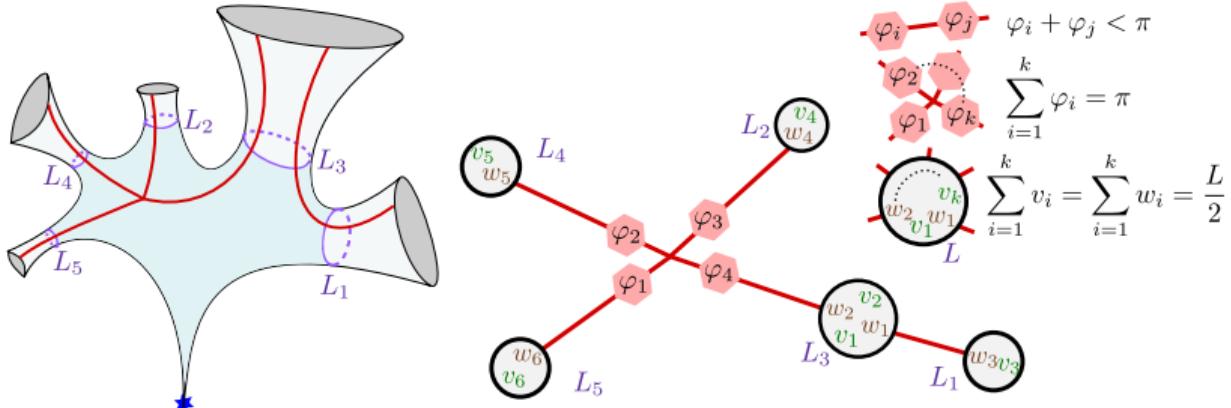
## Bijective result



- ▶ For tree  $t$  with  $n$  white vertices ( $\deg \geq 1$ ) and red vertices ( $\deg \geq 3$ ),

$$\mathcal{A}_t(L_1, \dots, L_n) = \{(\phi_i, v_i, w_i) : \phi_i > 0, v_i \geq 0, w_i > 0, \text{constraints above}\}.$$

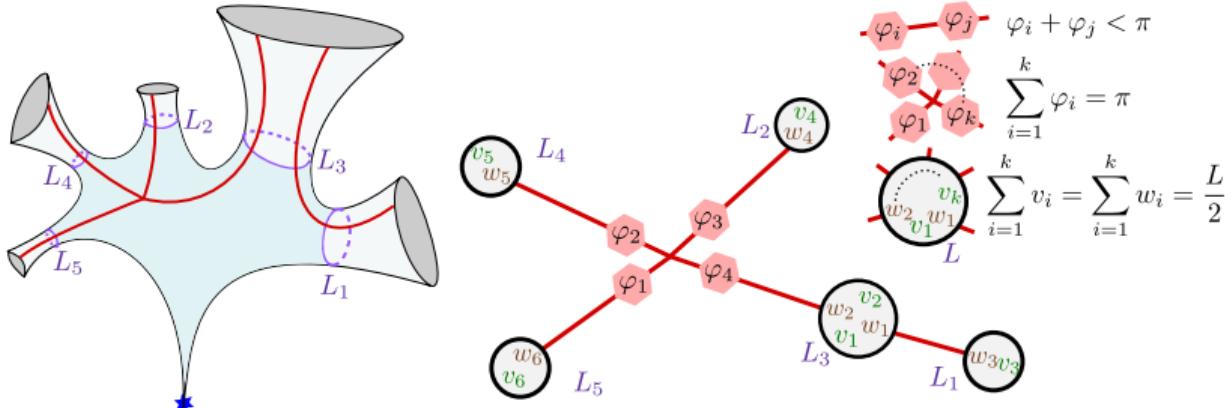
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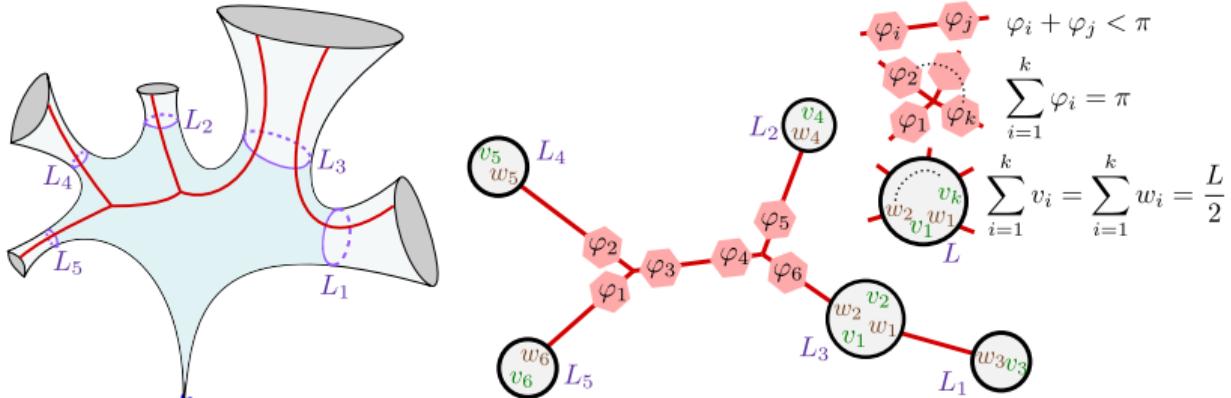
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Theorem (TB, Meeusen, Zonneveld, '23+)

This determines a bijection

$$\Phi : \mathcal{M}_{0,n+1}(0, L_1, \dots, L_n) \longleftrightarrow \bigsqcup_t \mathcal{A}_t(L_1, \dots, L_n).$$

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top-dim iff  $\deg(\bullet) = 3$

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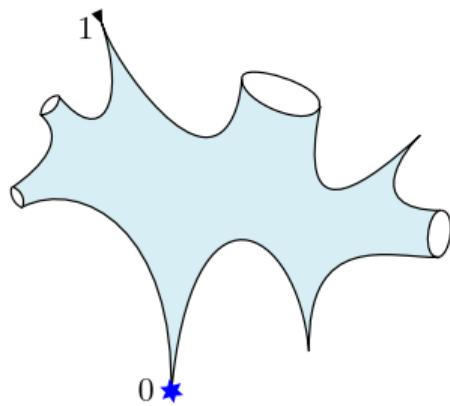
The push-forward of the WP volume is simply the Euclidean volume on the polytope  $\mathcal{A}_t \subset \mathbb{R}^{2n-4}$ ,

$$\Phi_* \mu_{\text{WP}} = \prod_{\circ} 2^{k-1} dw_1 dv_1 \cdots dw_{k-1} dv_{k-1} \prod_{\bullet} 2 d\phi_1 d\phi_2.$$

## WP volume generating function

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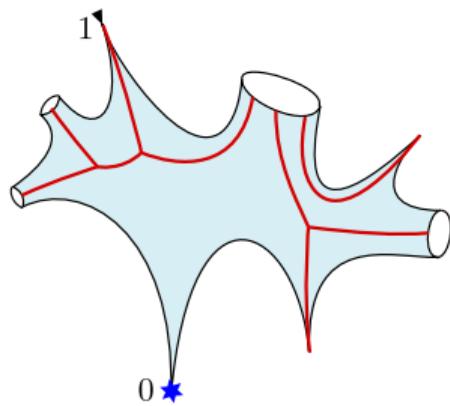
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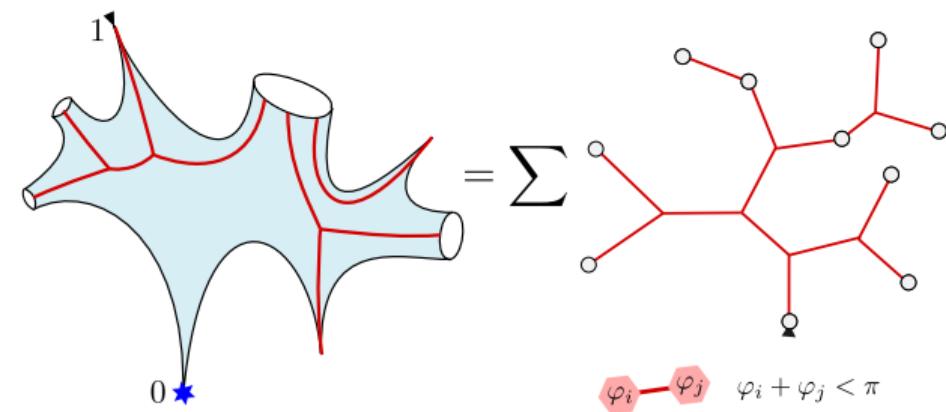
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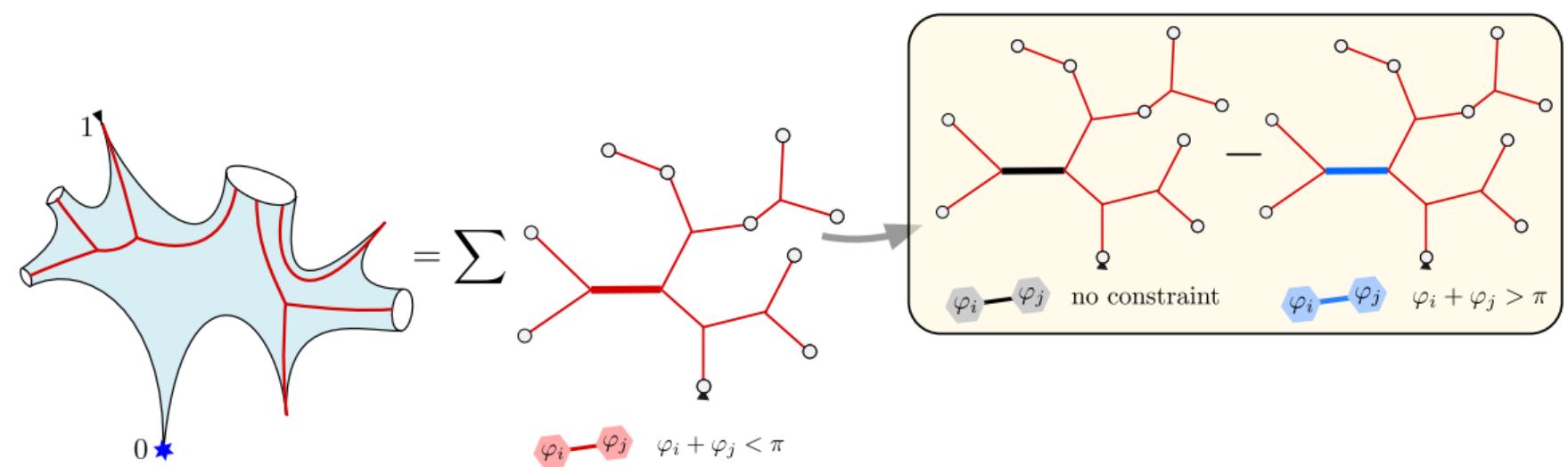
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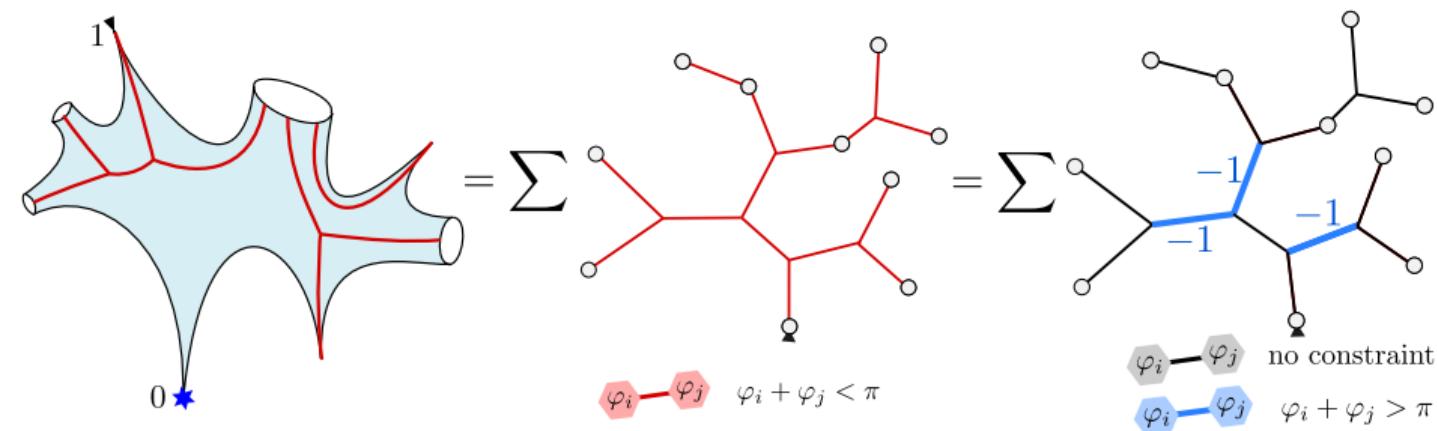
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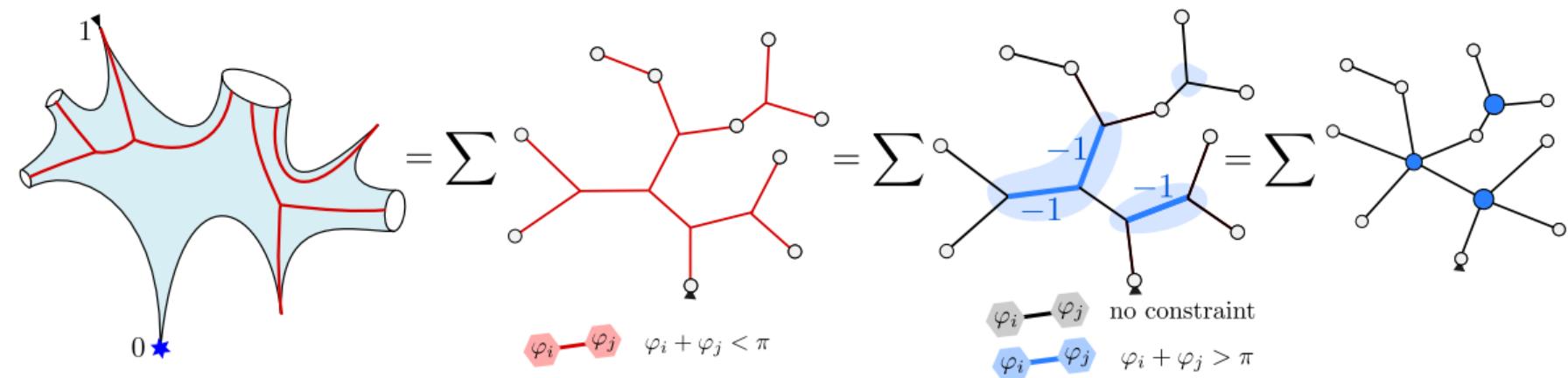
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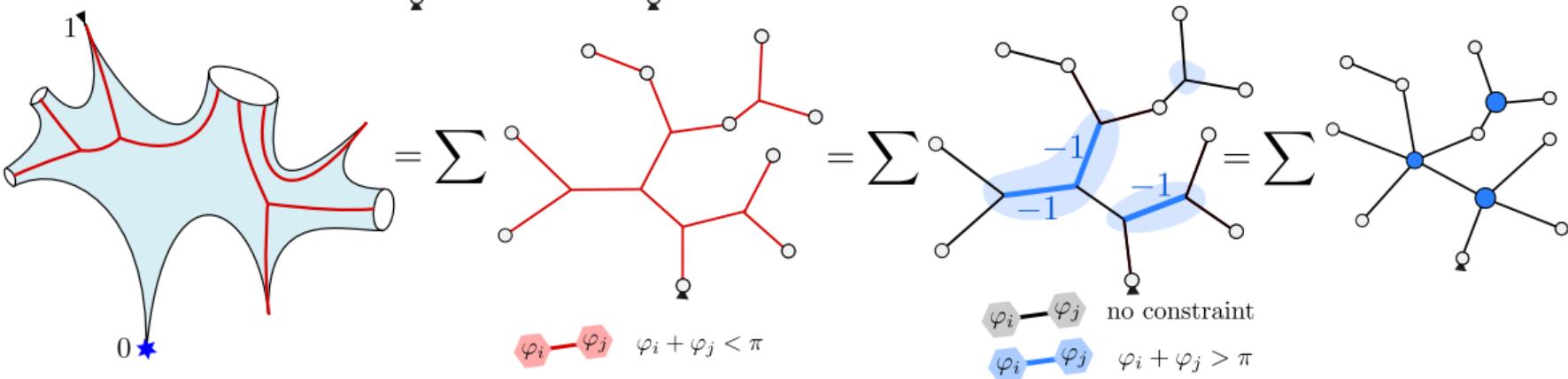
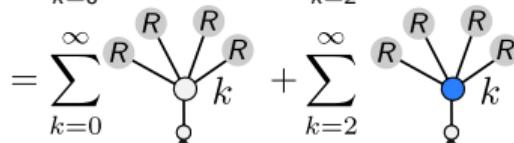
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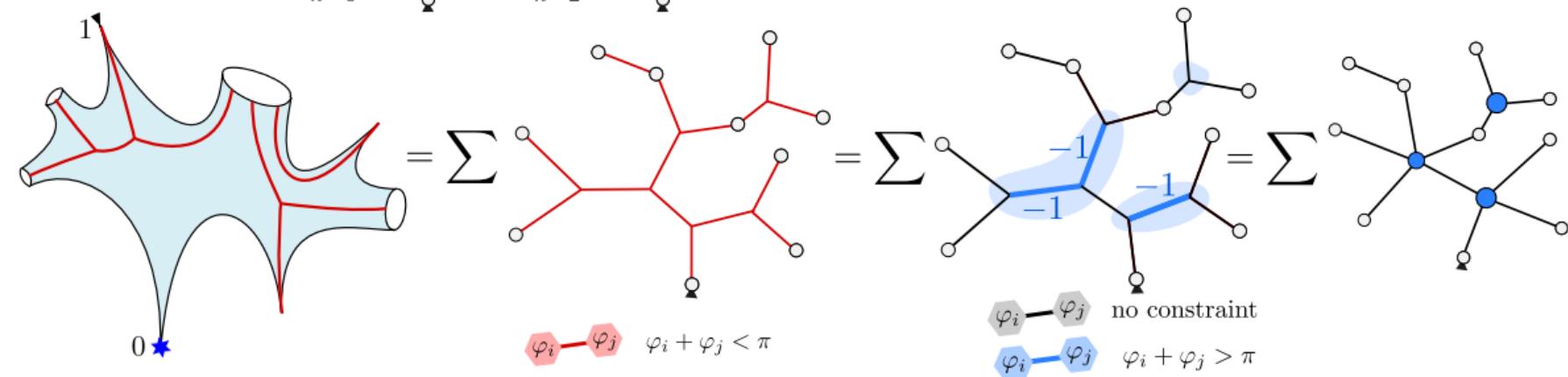
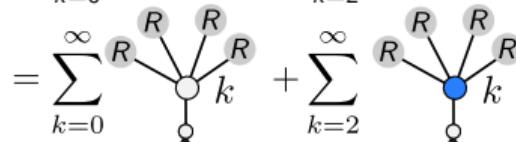
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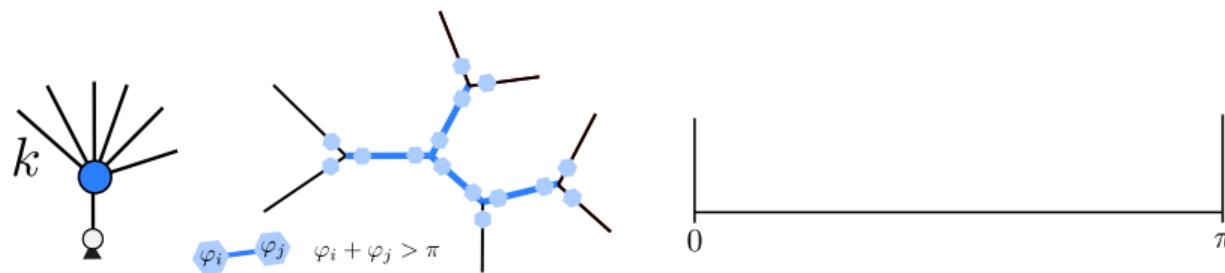
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## WP volume of blue vertices

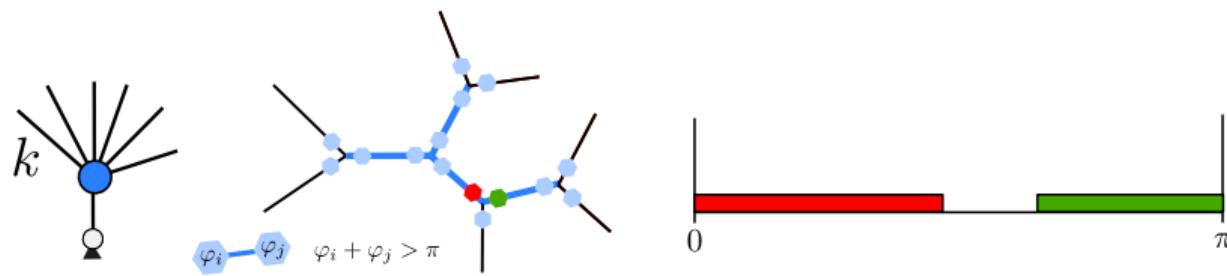
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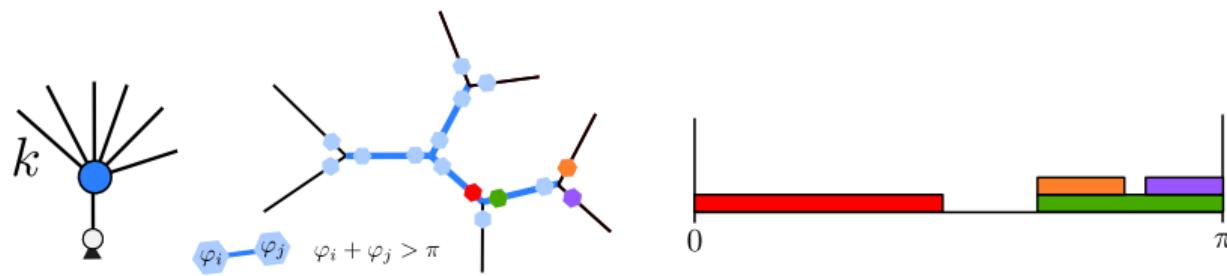
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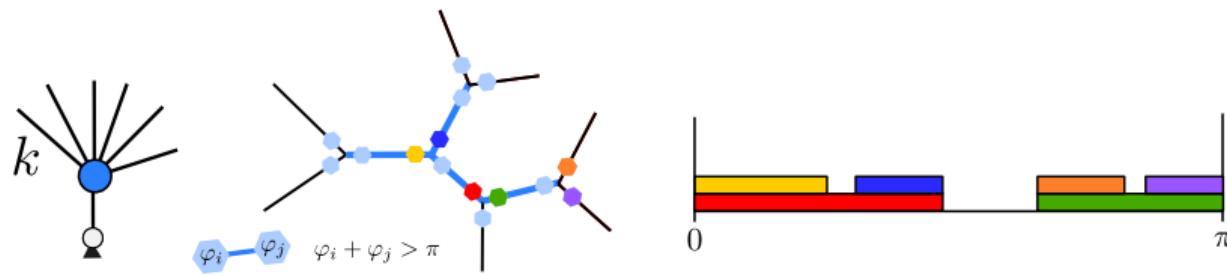
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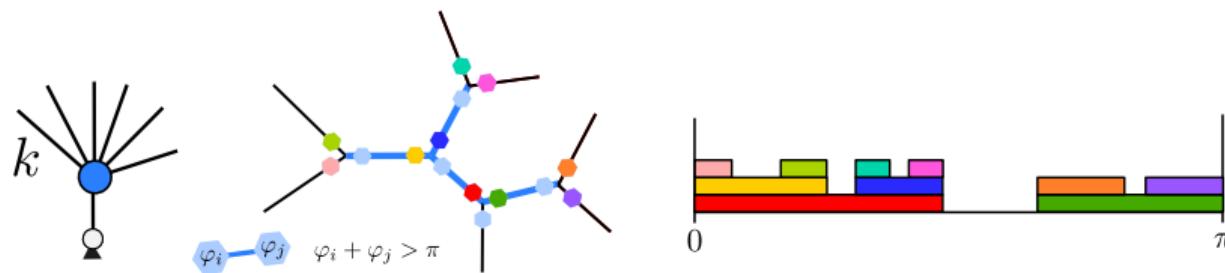
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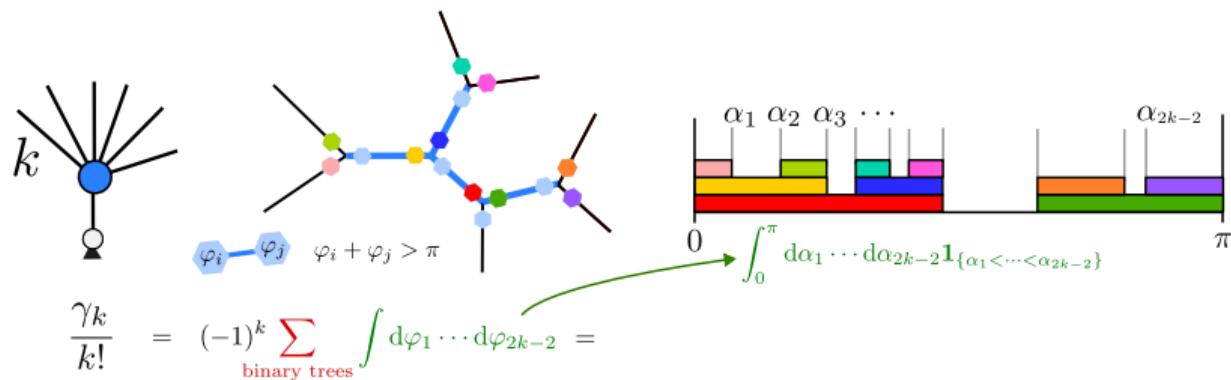
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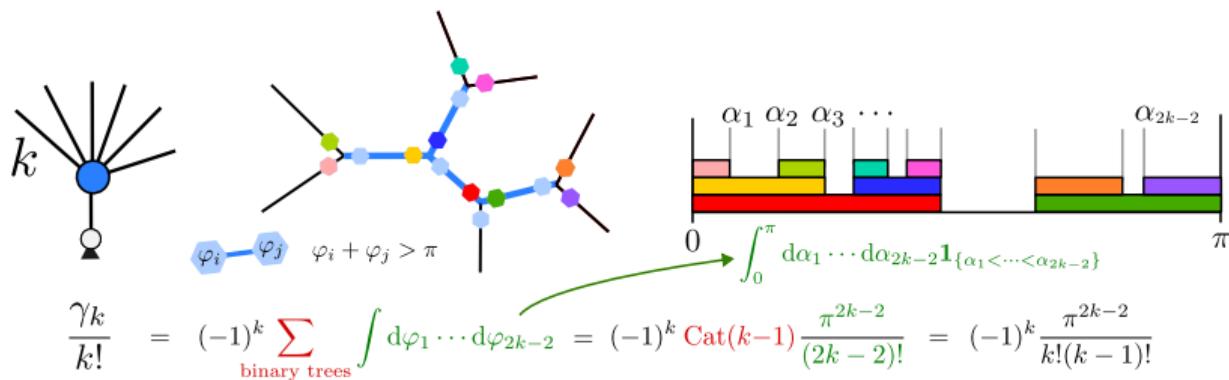
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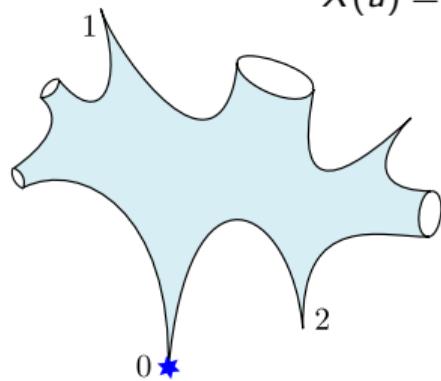
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## Not just volumes: geodesic distance control!

- ▶ Consider the distance-dependent generating function of triply-cusped surfaces

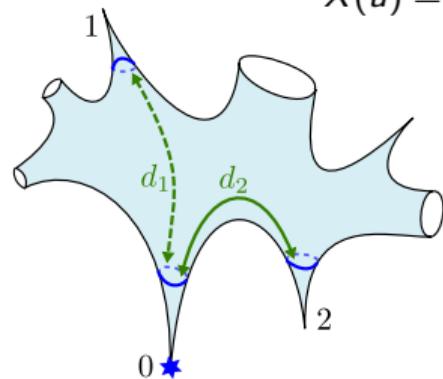
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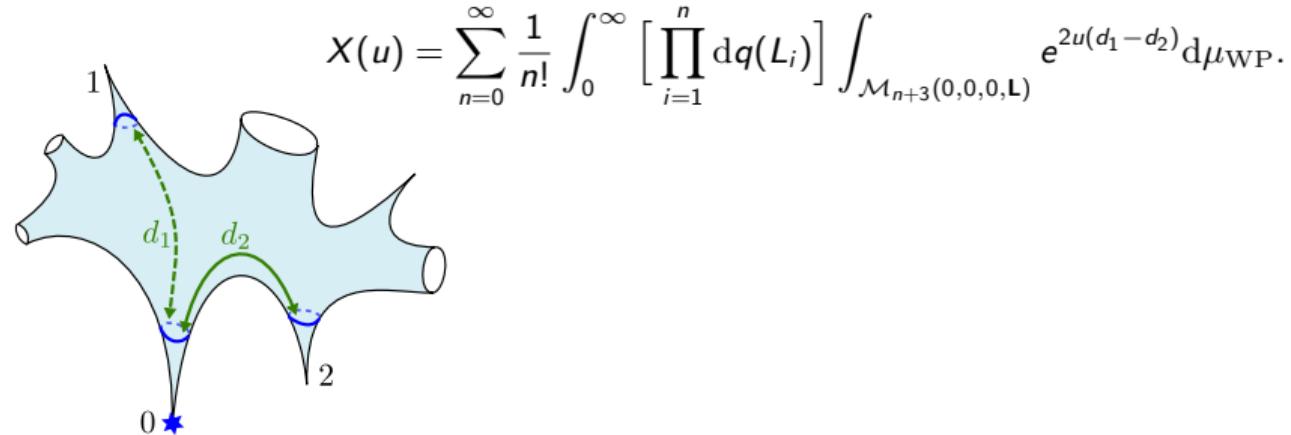
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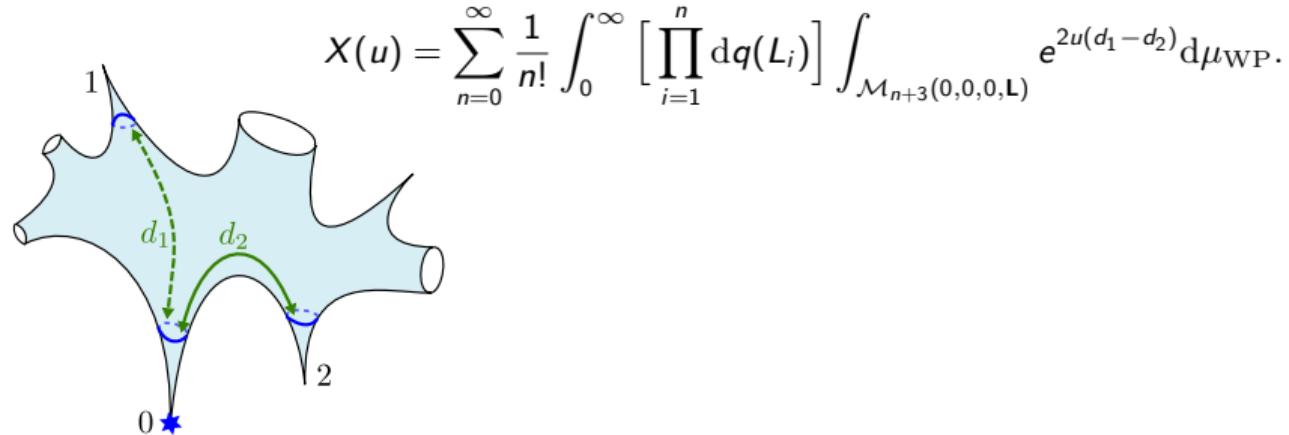


Theorem (TB, Meeusen, Zonneveld, '23+)

$$X(u) = \frac{\sin 2\pi u}{\pi y(u)}, \quad y(u) = [u^{\geq 0}] \frac{1}{\pi} \sin 2\pi z - \int_0^{\infty} dq(L) \frac{\cosh L z}{z}, \quad z = \sqrt{u^2 + 2R}$$

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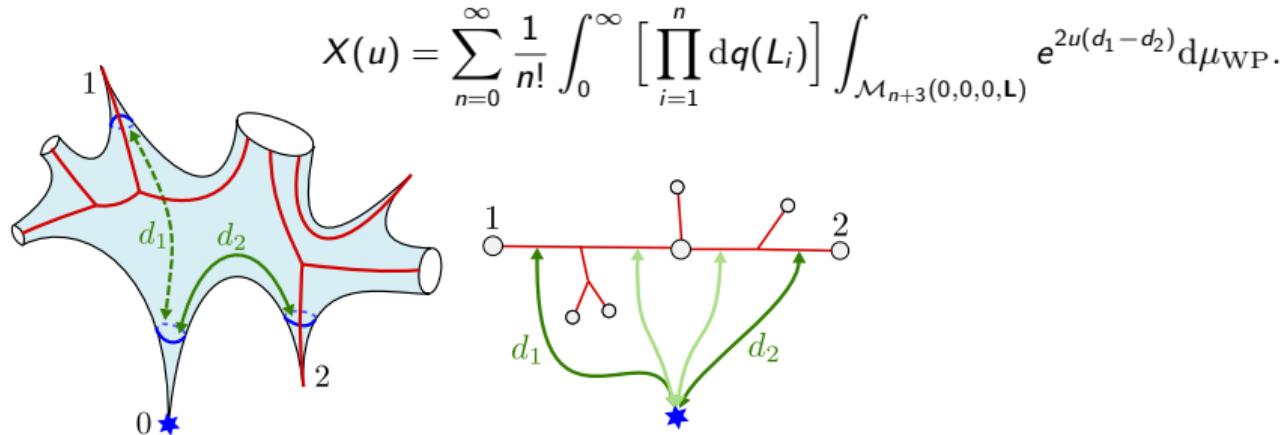


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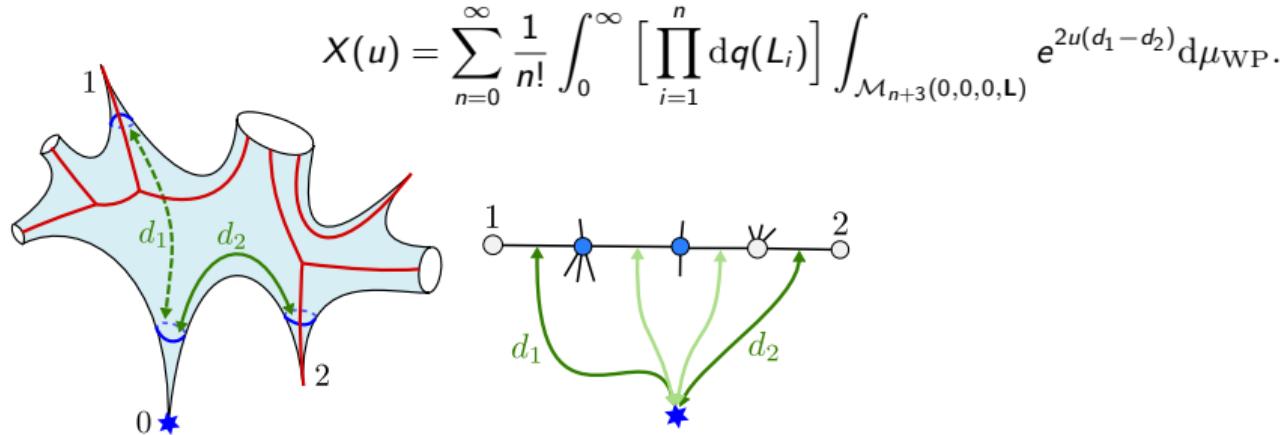


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The diagram illustrates a triply cusped surface on the left and its corresponding graph on the right. The surface has three cusps labeled 1, 2, and 0. Two distances,  $d_1$  and  $d_2$ , are indicated between the cusps. The graph on the right shows a horizontal line segment with vertices at 1, 0, 2. Two green curves, labeled  $d_1$  and  $d_2$ , connect the vertex 0 to the vertices 1 and 2 respectively. A blue dot is placed on the segment between 0 and 1, and another blue dot is placed between 1 and 2. A blue star is placed at the vertex 0.

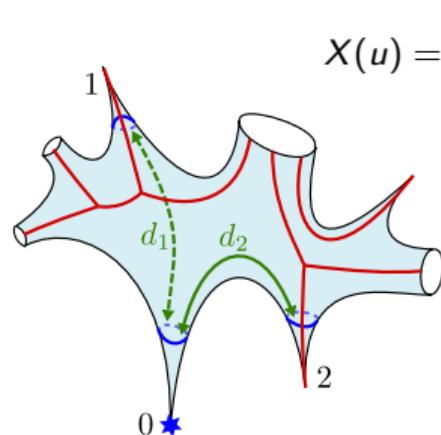
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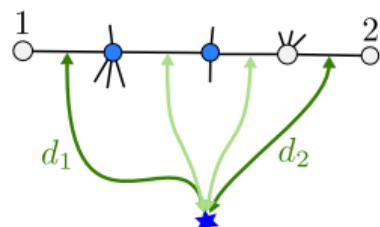
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## Not just volumes: geodesic distance control!

- ▶ Consider the distance-dependent generating function of triply-cusped surfaces



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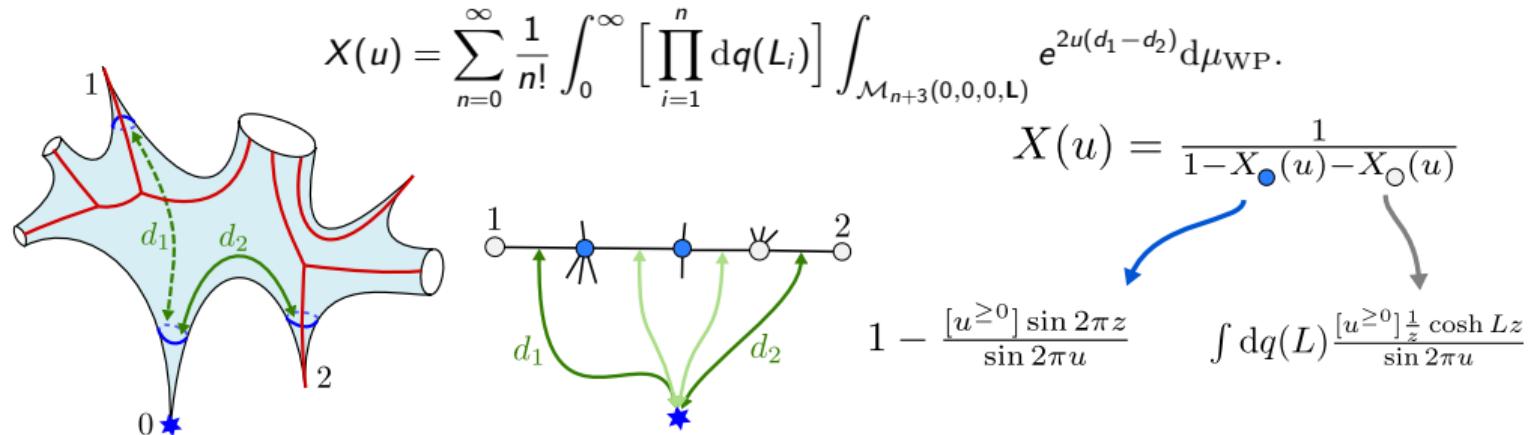
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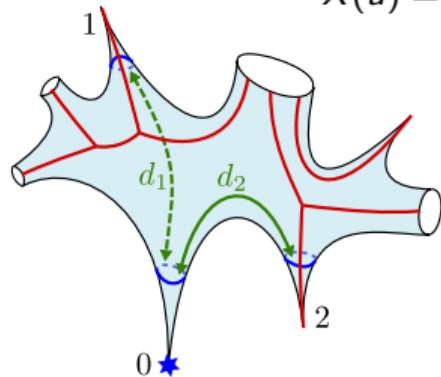
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The diagram shows a horizontal line representing a chain. Site 1 is at the left end, represented by a circle with a vertical line. Site 2 is at the right end, also represented by a circle with a vertical line. Between them is a central site, represented by a blue circle with a vertical line. Green arrows point from both site 1 and site 2 towards the central site. Below the line, a green arrow points upwards from a blue star at the bottom center towards the central site.

$$X(u) = \frac{1}{1 - X_{\bullet}(u) - X_{\circlearrowleft}(u)}$$

$$1 - \frac{[u^{\geq 0}] \sin 2\pi z}{\sin 2\pi u} \quad \int dq(L) \frac{[u^{\geq 0}]_z^{\frac{1}{2}} \cosh Lz}{\sin 2\pi u}$$

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  - ▶ Singularity analysis:  $d_1 - d_2 \approx n^{1/4}$  in Boltzmann hyperbolic sphere for  $n$  large. Same universality class as Boltzmann planar map?

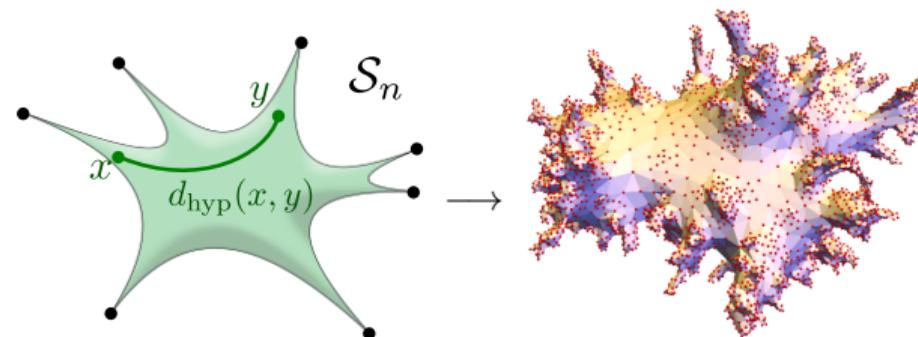
## Control on hyperbolic distances

- ▶ In the case of only cusps,  $q(L) = x\delta(L)$ , this is indeed true:

Theorem (TB, Curien, '23+)

If  $S_n \in \mathcal{M}_{0,n}(0)$  is sampled with probability density  $\mu_{WP}/V_{0,n}(0)$ , then we have the convergence in distribution of the random metric space in the Gromov–Prokhorov topology

$$\left( S_n, \frac{d_{\text{hyp}}}{c n^{1/4}} \right) \xrightarrow[n \rightarrow \infty]{(\text{d})} \text{Brownian sphere}, \quad c = 2.339 \dots$$



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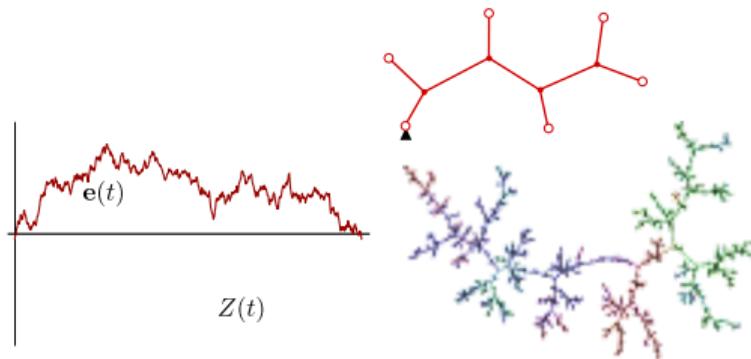
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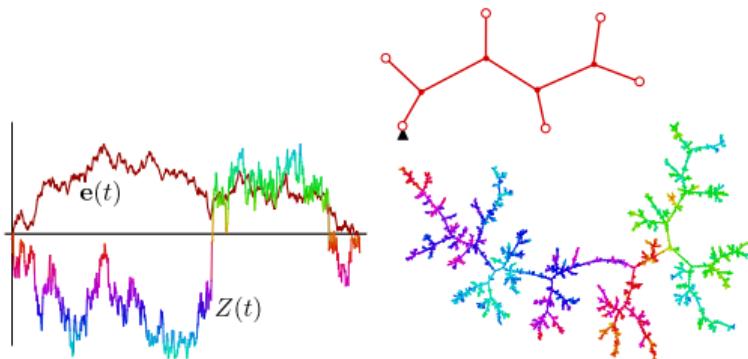
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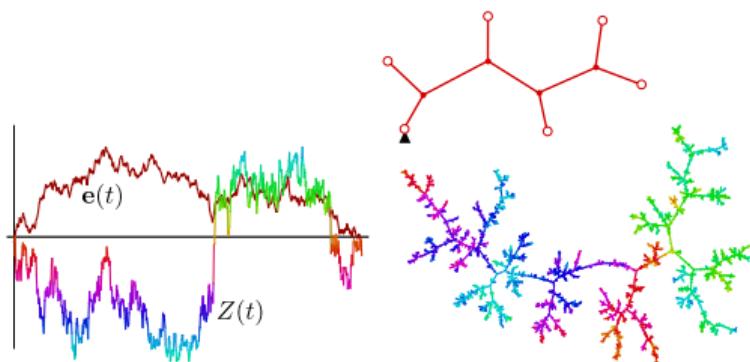
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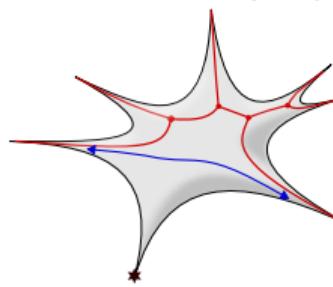
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bound on distances  
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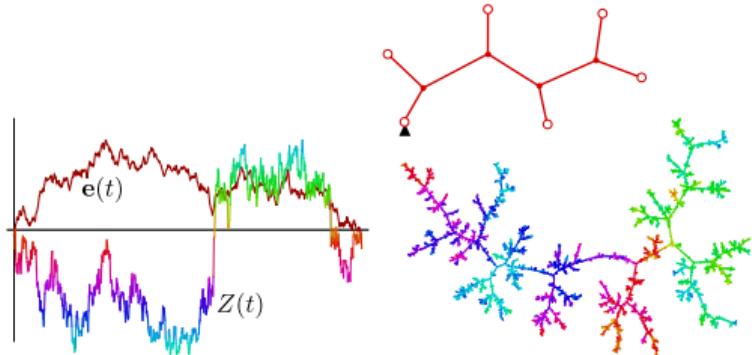
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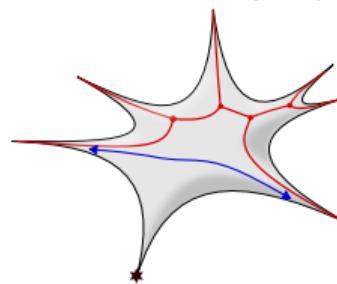
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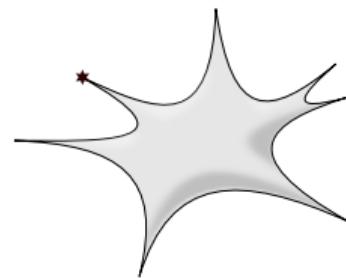
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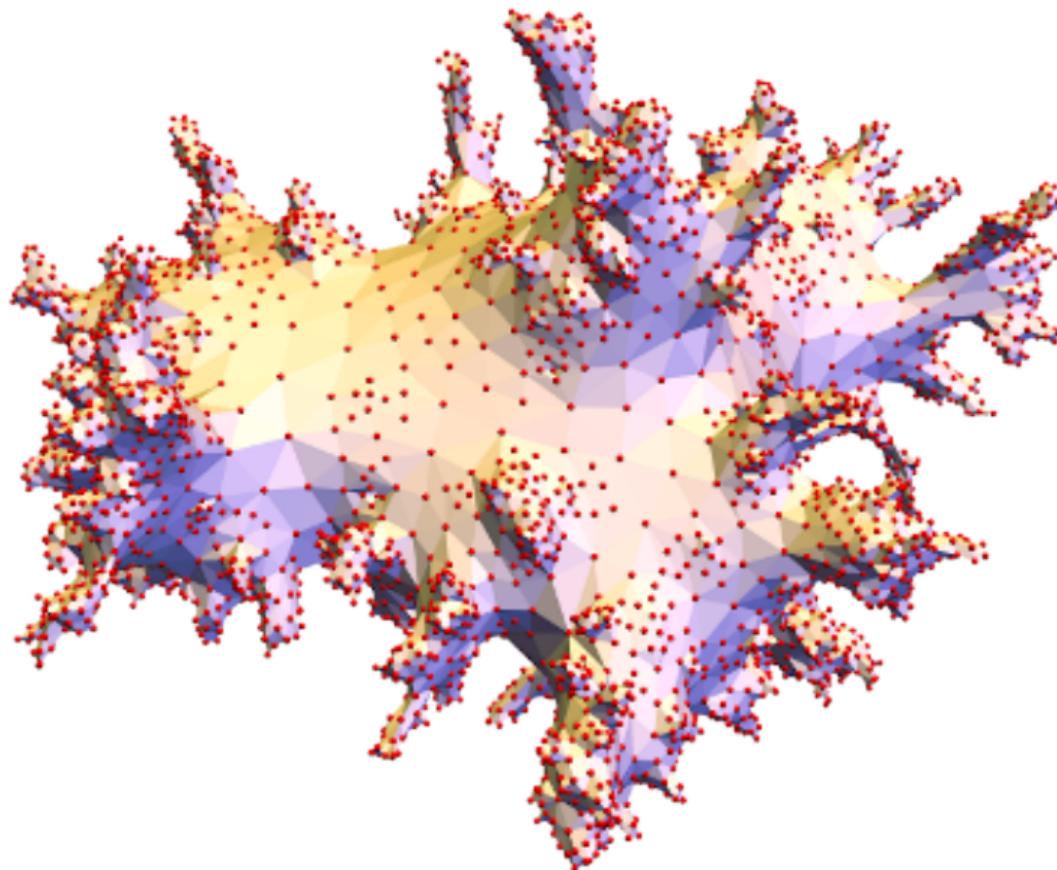


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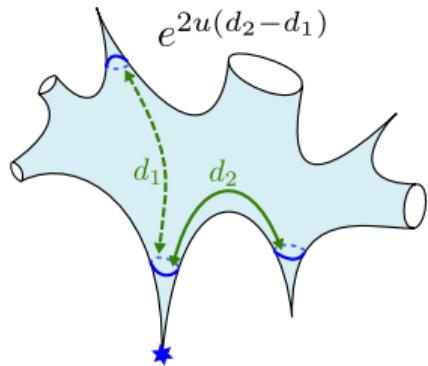
invariance under  
change of origin



Thanks for your attention!



## Topological recursion



$$X(u) = \frac{\sin 2\pi u}{\pi y(u)}, \quad z = \sqrt{u^2 + 2R}$$

$$y(u) = [u^{\geq 0}] \frac{1}{\pi} \sin 2\pi z - \int_0^\infty dq(L) \frac{\cosh Lz}{z},$$

Theorem (TB, Zonneveld, '23+)

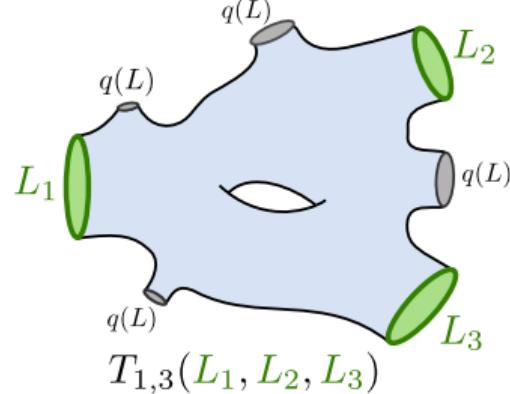
The invariants  $\omega_{g,n}(z)$  of the curve  $(x(u) = u^2, y(u))$  with initial condition  $\omega_{0,2}(z) = (z_1 - z_2)^2$  and topological recursion

$$\omega_{g,n}(z) = \text{Res}_{u \rightarrow 0} \frac{1}{(z_1^2 - u^2)y(u)} \left[ \omega_{g-1,n+1}(u, -u, z_{\widehat{\{1\}}}) + \sum_{\substack{g_1 + g_2 = g \\ I \sqcup J = \{2, \dots, n\}}} \omega_{g_1,n_1}(u, z_I) \omega_{g_2,n_2}(-u, z_J) \right]$$

give the Laplace transforms of “*Tight Weil–Petersson volumes*”  $T_{g,n}(L_1, \dots, L_n)$ ,

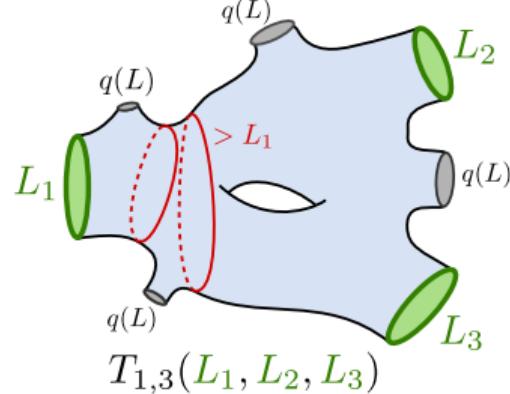
$$\omega_{g,n}(z) = \int_0^\infty dL_1 L_1 e^{-z_1 L_1} \dots \int_0^\infty dL_n L_n e^{-z_n L_n} T_{g,n}(L_1, \dots, L_n).$$

## Tight Weil-Petersson volumes



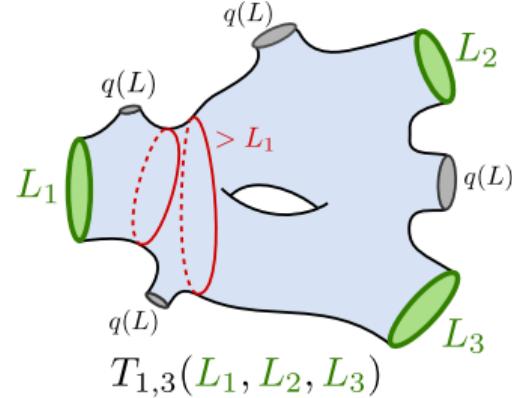
$$T_{g,n}(\mathbf{L}) = \sum_{p=0}^{\infty} \frac{1}{p!} \int dq(L_{n+1}) \int dq(L_{n+p}) \int_{\mathcal{M}_{g,n+p}(\mathbf{L}, \mathbf{L})} d\mu_{\text{WP}} \mathbf{1}_{\{\text{tight}\}}$$

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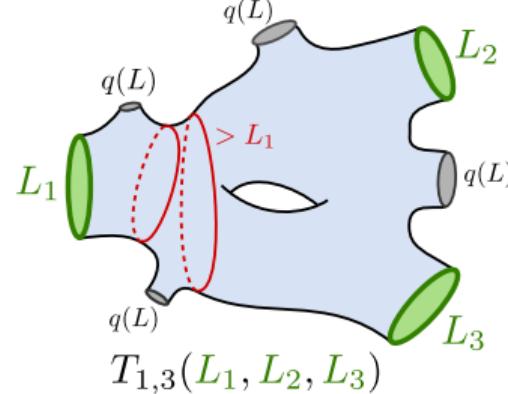


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Topological recursion pictorially:

The diagram illustrates the topological recursion relation for a surface with boundary components  $L_1$ ,  $L_2$ , and  $L_3$ . The equation shows the surface  $L_1$  as equal to a sum of two terms: one where a green shaded region is added to the surface, and another where a blue shaded region is added. The regions are separated by a dashed line.

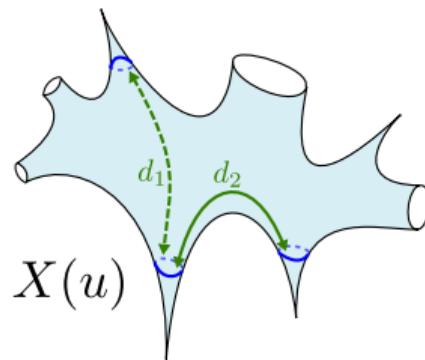
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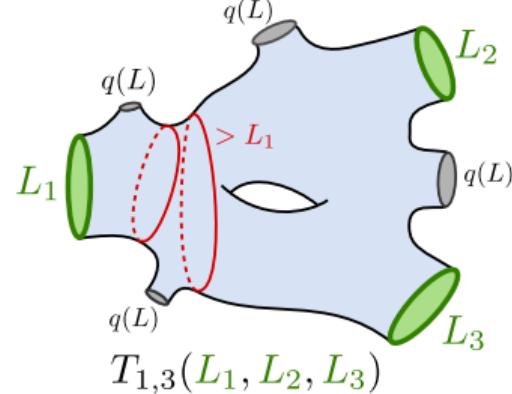
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Topological recursion pictorially:

The diagram illustrates the topological recursion equation. It shows a surface  $L_1, L_2, L_3$  on the left, followed by an equals sign, then two terms separated by a plus sign. The first term shows a surface  $L_1, L_2, L_3$  with a green shaded region. The second term shows a surface  $L_1, L_2, L_3$  with a blue shaded region.



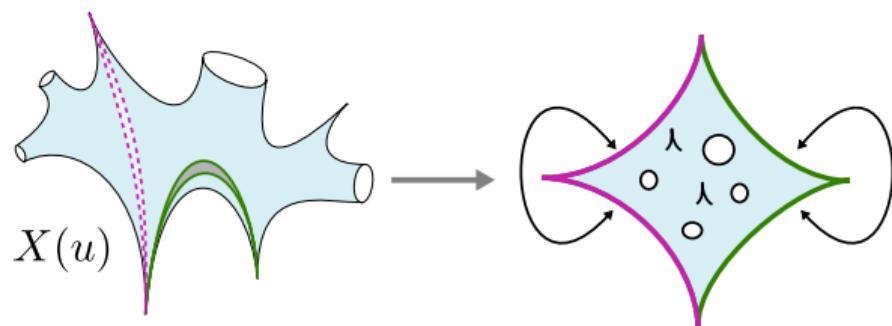
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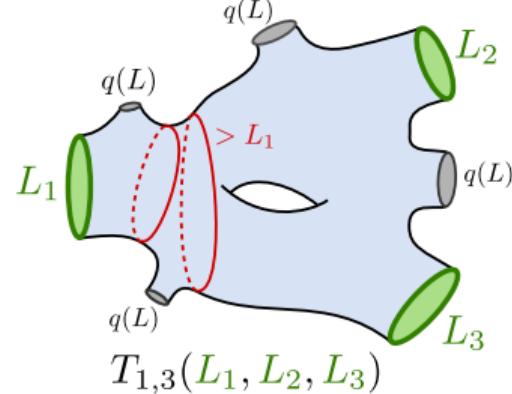
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Topological recursion pictorially:

$$\begin{array}{c} L_1 \\ \text{---} \\ L_2 \\ \text{---} \\ L_3 \end{array} = \sum \begin{array}{c} L_1 \\ \text{---} \\ L_2 \\ \text{---} \\ L_3 \end{array} + \sum \begin{array}{c} L_1 \\ \text{---} \\ L_2 \\ \text{---} \\ L_3 \end{array}$$



## Tight Weil-Petersson volumes



$$T_{1,3}(\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3)$$

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Topological recursion pictorially:

$$\begin{array}{c} \text{Diagram of a surface with boundary components } L_1, L_2, \text{ and } L_3. \\ = \sum \text{Diagram where a green shaded region covers the surface, with boundary components } L_1, L_2, \text{ and } L_3. \\ + \sum \text{Diagram where a green shaded region covers the surface, with boundary components } L_1, L_2, \text{ and } L_3. \end{array}$$

