Supersymmetric Quantum Mechanics and Morse Theory

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Solution to Homework 3: $d = 0$ Supersymmetry

1 Linking number

Introduction: For pair of loops (curve with no boundary) C_1 and C_2, given in parametric form

$$C_1 = S^1 \to \mathbb{R}^3 : t \mapsto \gamma_1^i(t), \quad C_2 = S^1 \to \mathbb{R}^3 : t \mapsto \gamma_2^i(t) \tag{1.1}$$

the formula for the linking number is well known

$$\mathrm{lk}(C_1, C_2) = \frac{1}{4\pi} \int_0^{2\pi} dt_1 \int_0^{2\pi} dt_2 \sum_{ijk} \frac{\epsilon_{ijk} \dot{\gamma}_1^i(t_1) \dot{\gamma}_2^j(t_2) (\gamma_1(t_1) - \gamma_2(t_2))^k}{|\gamma_1(t_1) - \gamma_2(t_2)|^3}. \tag{1.2}$$

On the lecture we briefly discussed the relation between the linking number and intersection number. The goal of this problem is to derive the formula above from the intersection theory.

Construction: The Poincare lemma for $\mathbb{R}^3$ implies that

$$H_1(\mathbb{R}^3) = H^1(\mathbb{R}^3) = 0, \tag{1.3}$$

so for any loop $C \subset \mathbb{R}^3$ there exists a surface $S \subset \mathbb{R}^3$ such that

$$\partial S = C. \tag{1.4}$$

For two loops C_1 and C_2 in $\mathbb{R}^3$ we can define linking number via the intersection number

$$\mathrm{lk}(C_1, C_2) = C_1 \cdot S, \tag{1.5}$$

where S is surface, homotopic to a disc, such that $\partial S = C_2$. We can use the integral formula for the intersection number

$$C \cdot S = \int_{\mathbb{R}^3} \eta_C \wedge \eta_S. \tag{1.6}$$

The 1-form η_C is defined via

$$\int_C \omega = \int_{\mathbb{R}^3} \eta_C \wedge \omega, \quad \forall \omega \in \Omega^1(\mathbb{R}^3). \tag{1.7}$$

The Poincare dual form η_C for loop C is trivial in cohomology

$$[\eta_C] = 0 \in H^2(\mathbb{R}^3) = 0, \tag{1.8}$$

hence we can write it as

$$\eta_C = d\eta_S. \tag{1.9}$$

Formally, the 1-form η_S is given by

$$\eta_S = d^{-1}\eta_C, \tag{1.10}$$

but the inverse of external derivative is not a differential operator on forms. We can use homotopy to construct differential operator version of d^{-1}

$$\eta_C = (1-\Pi_0)\eta_C = (dK + Kd)\eta_C = dK\eta_C \implies \eta_S = K\eta_C, \tag{1.11}$$

where homotopy

$$K = \int_0^\infty ds \;\; d^* e^{-s\Delta}. \tag{1.12}$$

1. (*10 points*) Evaluate the Poincare dual η_C for the closed loop given in parametric form

$$C = \gamma : S^1 \to \mathbb{R} : t \mapsto \gamma(t). \tag{1.13}$$

Solution: The Poincare dual η_C to curve $C \subset \mathbb{R}^3$

- should be 2-form on $\mathbb{R}^3$,
- should have support on C, which for parametrically given C can be schematically written as
$$\eta_C \propto \int_0^{2\pi} dt \;\; \delta^3(x-\gamma(t)), \tag{1.14}$$
- should be invariant under change of coordinates on $\mathbb{R}^3$ i.e. all indices should be properly constructed

what naturally leads us to the ansatz for η_C in the form

$$\eta_C = \frac{1}{2}\int_0^{2\pi} dt\, \delta^3(x-\gamma(t)) \sum_{ijk} \epsilon_{ijk}\dot{\gamma}^i dx^j \wedge dx^k \tag{1.15}$$

with numerical coefficient being restored from definition of Poincare dual. A generic generic 1-form in the same coordinates

$$\omega = \omega_i(x)dx^i \in \Omega^1(\mathbb{R}^3) \tag{1.16}$$

integrated over C

$$\int_C \omega = \int_{S^1} \gamma^* \omega = \int_0^{2\pi} dt \ \sum \omega(\gamma(t))_i \dot{\gamma}^i(t) \tag{1.17}$$

and integrated over $\mathbb{R}^3$

$$\begin{aligned}
\int_{\mathbb{R}^3} \eta_C \wedge \omega &= \int_0^{2\pi} dt \int_{\mathbb{R}^3} \delta^3(x-\gamma(t)) \sum_{ijk} \frac{1}{2} \epsilon_{ijk} \dot{\gamma}^i dx^j \wedge dx^k \wedge \sum \omega_l(x) dx^l \\
&= \int_0^{2\pi} dt \int_{\mathbb{R}^3} dx^1 \wedge dx^2 \wedge dx^3 \delta^3(x-\gamma(t)) \sum_{il} \delta_i^l \dot{\gamma}^i \omega_l(x) \\
&= \int_0^{2\pi} dt \sum_i \dot{\gamma}^i \omega_i(\gamma(t)) = \int_C \omega
\end{aligned} \tag{1.18}$$

where we used

$$dx^j \wedge dx^k \wedge dx^l = \epsilon^{jkl} dx^1 \wedge dx^2 \wedge dx^3 \in \Omega^3(\mathbb{R}^3) = \Omega^{top}(\mathbb{R}^3). \tag{1.19}$$

2. (*10 points*) Evaluate the $d^* \eta_C$ for η_C from part 1.

Solution: By definition

$$d^* = (-1)^{np+n+1} * d * : \ \ \Omega^p(M) \to \Omega^{p-1}(M). \tag{1.20}$$

acting on $\Omega^2(\mathbb{R}^3)$ simplifies into

$$d^* = *d* : \ \ \Omega^2(\mathbb{R}^3) \to \Omega^1(\mathbb{R}^3). \tag{1.21}$$

The evaluation gives us

$$\begin{aligned}
d^* \eta_C = *d * \eta_C &= \int dt \ * d * \frac{1}{2} \sum_{ijk} \delta^3(x-\gamma(t)) \epsilon_{ijk} \dot{\gamma}^i dx^j \wedge dx^k \\
&= \int dt \ * d \sum_{ij} \delta^3(x-\gamma(t)) \delta_{ij} \dot{\gamma}^i dx^j \\
&= \int dt \ * \sum_{ijk} \partial_k \delta^3(x-\gamma(t)) \delta_{ij} \dot{\gamma}^i dx^k \wedge dx^j \\
&= \int dt \ \sum_{ijk} \partial_i \delta^3(x-\gamma(t)) \epsilon^{ijk} \dot{\gamma}^j \delta_{kl} dx^l
\end{aligned} \tag{1.22}$$

3. (*10 points*) Evaluate the Hodge-Laplacian $\Delta = dd^* + d^*d$ for 1-form on $\mathbb{R}^3$ written in components

$$\omega = \sum_{i=1}^{3} \omega_i(x)dx^i \tag{1.23}$$

Hint: You can use either of 3 methods:

- Straightforward evaluation in terms of differential forms.
- In HW 2 we related d and d^* on $\mathbb{R}^3$ to divergence, gradient and curl, so we can use vector calculus in 3d to evaluate Hodge-Laplacian.
- In HW 2 we related d and d^* on $\mathbb{R}^3$ to certain differential operators on $\Pi T\mathbb{R}^3 = \mathbb{R}^{3|3}$, so we can use differential operators representation to evaluate Hodge-Laplacian.

4. (*10 points*) Use your answers for $d^*\eta_C$ and Hodge-Laplacian action on forms to evaluate the $\eta_S = K\eta_C$. Use integral representation for δ-function in η_C

$$\delta^3(x) = \prod_{k=1}^{3} \delta(x^k) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} d^3p \; e^{i\sum p_k x^k} \tag{1.24}$$

Solution: The integral representation for $d^*\eta_C$

$$d^*\eta_C = \frac{i}{(2\pi)^3} \int ds \int d^3p \int_0^{2\pi} dt \; e^{ip_l(x-\gamma(t))^l} \sum_{ijk} \epsilon_{ijk}\dot{\gamma}^i p^j dx^k \tag{1.25}$$

while the exponent of Laplacian acting on components of 1-form

$$e^{-s\Delta} \exp\left(i\sum p_l(x-\gamma(t))^l\right) = \exp\left(i\sum p_l(x-\gamma(t))^l - s\sum p_l\delta^{lm}p_m\right) \tag{1.26}$$

Using simplified physics notations the same formula is much more compact

$$e^{-s\Delta}e^{ip(x-\gamma(t))} = e^{ip(x-\gamma(t))-sp^2} \tag{1.27}$$

The action of homotopy

$$\begin{aligned} K\eta_C &= \int_0^\infty ds \; e^{-s\Delta} d^*\eta_S \\ &= \frac{i}{(2\pi)^3} \int_0^\infty ds \int d^3p \int_0^{2\pi} dt \; e^{ip(x-\gamma(t))-sp^2} \sum_{ijk} \epsilon_{ijk}\dot{\gamma}^i(t)p^j dx^k \end{aligned}$$

5. (*10 points*) Evaluate the intersection number

$$C \cdot S = \int_{\mathbb{R}^3} \eta_C \wedge \eta_S \tag{1.28}$$

by performing the $\mathbb{R}^3$ integral first, p-integral second and s-integral last.

Solution: The linking number

$$\begin{aligned}
&\mathrm{lk}(C_1, C_2) = C_1 \cdot S = \int_{\mathbb{R}^3} \eta_{C_1} \wedge K\eta_{C_2} = \frac{i}{(2\pi)^3} \int_0^{2\pi} dt_1 \int_0^{2\pi} dt_2 \int_0^{\infty} ds \int d^3p \\
&\int_{\mathbb{R}^3} \delta^3(x - \gamma_1(t_1)) \sum_{ijk} \epsilon_{ijk} \dot{\gamma}_1^i dx^j \wedge dx^k \wedge e^{ip(x-\gamma_2(t_2)) - sp^2} \sum_{ijk} \epsilon_{i'j'k'} \dot{\gamma}_2^{i'} p^{j'} dx^{k'} \\
&= \frac{i}{(2\pi)^3} \int_0^{2\pi} dt_1 \int_0^{2\pi} dt_2 \int_0^{\infty} ds \int d^3p e^{ip(\gamma_1(t_1)-\gamma_2(t_2)) - sp^2} \sum_{ijk} \epsilon_{ijk} \dot{\gamma}_1^i \dot{\gamma}_2^j p^k \\
&= \frac{\pi^{3/2}}{(2\pi)^3} \int_0^{2\pi} dt_1 \int_0^{2\pi} dt_2 \int_0^{\infty} \frac{ds}{s^{5/2}} e^{-\frac{1}{4s}|\gamma_1(t_1)-\gamma_2(t_2)|^2} \sum_{ijk} \epsilon_{ijk} \dot{\gamma}_1^i \dot{\gamma}_2^j (\gamma_1(t_1) - \gamma_2(t_2))^k \\
&= \frac{\pi^{3/2}}{(2\pi)^3} \int_0^{2\pi} dt_1 \int_0^{2\pi} dt_2 \int_0^{\infty} \frac{2(4\tau^2)^{5/2} d\tau}{4\tau^3} e^{-\tau^2|\gamma_1(t_1)-\gamma_2(t_2)|^2} \sum_{ijk} \epsilon_{ijk} \dot{\gamma}_1^i \dot{\gamma}_2^j (\gamma_1(t_1) - \gamma_2(t_2))^k \\
&= \frac{1}{2\pi^{3/2}} \int_0^{2\pi} dt_1 \int_0^{2\pi} dt_2 \sum_{ijk} \epsilon_{ijk} \dot{\gamma}_1^i \dot{\gamma}_2^j (\gamma_1(t_1) - \gamma_2(t_2))^k \int_0^{\infty} d\tau \; \tau^2 e^{-\tau^2|\gamma_1(t_1)-\gamma_2(t_2)|^2} \\
&= \frac{1}{4\pi} \int_0^{2\pi} dt_1 \int_0^{2\pi} dt_2 \sum_{ijk} \frac{\epsilon_{ijk} \dot{\gamma}_1^i \dot{\gamma}_2^j (\gamma_1(t_1) - \gamma_2(t_2))^k}{|\gamma_1(t_1) - \gamma_2(t_2)|^3}.
\end{aligned} \tag{1.29}$$

2 $d = 0$ $N = 4$ supersymmetry

Let $W(z)$ be the holomorphic ($\bar{z}$-independent) polynomial with complex coefficients. Let us consider a integral over the complex plane

$$Z_W = \frac{1}{2\pi i} \int_{\mathbb{C}} dz d\bar{z} \;\; e^{-W(z)\overline{W(z)}} \partial_z W(z) \overline{\partial_z W(z)} \tag{2.1}$$

We can introduce four Grassmann-odd variables $\psi^z, \bar{\psi}^z$, $\psi^{\bar{z}}$ and $\bar{\psi}^{\bar{z}}$ to rewrite the integral as partition function

$$\begin{aligned} Z_W &= \frac{i}{\pi} \int d^4\psi dz d\bar{z} \;\; e^{-S(z,\psi)}, \\ S(z,\psi) &= W(z)\overline{W(z)} - \partial_z W \psi^z \bar{\psi}^z - \overline{\partial_z W(z)} \psi^{\bar{z}} \bar{\psi}^{\bar{z}} \end{aligned} \tag{2.2}$$

1. (*10 points*) Show that the action $S(\psi, z)$ is invariant under the four Grassmann-odd symmetries.

   Hint: You can use one copy of $N = 2$ transformation for z and another copy for $\bar{z}$ so that

$$\begin{aligned} \delta z &= \epsilon^z \bar{\psi}^z + \bar{\epsilon}^z \psi^z \\ \delta \bar{z} &= \epsilon^{\bar{z}} \bar{\psi}^{\bar{z}} + \bar{\epsilon}^{\bar{z}} \psi^{\bar{z}} \end{aligned} \tag{2.3}$$

   Solution: The transformations for ψ follow from invariance of the action. The ϵ^z- and $\bar{\epsilon}^z$-transformations, do not affect $\bar{z}$, so they should not affect the $\psi^{\bar{z}}$ and $\bar{\psi}^{\bar{z}}$ as well. The ϵ^z- and $\bar{\epsilon}^z$- transformation of the action is of the form

$$\begin{aligned} \delta S &= \delta z \partial_z W \overline{W} - \delta z \partial_z^2 W \psi^z \bar{\psi}^z - \partial_z W \delta\psi^z \bar{\psi}^z - \partial_z W \psi^z \delta\bar{\psi}^z \\ &= (\epsilon^z \bar{\psi}^z + \bar{\epsilon}^z \psi^z) \partial_z W(z) \overline{W} - (\epsilon^z \bar{\psi}^z + \bar{\epsilon}^z \psi^z) \partial_z^2 W \psi^z \bar{\psi}^z - \partial_z W \delta\psi^z \bar{\psi}^z - \partial_z W \psi^z \delta\bar{\psi}^z \\ &= \partial_z W \epsilon^z \overline{W} \bar{\psi}^z + \partial_z W \bar{\epsilon}^z \overline{W} \psi^z - \partial_z W \delta\psi^z \bar{\psi}^z + \partial_z W \delta\bar{\psi}^z \psi^z \\ &= \partial_z W (\epsilon^z \overline{W} - \delta\psi^z) \bar{\psi}^z + \partial_z W (\bar{\epsilon}^z \overline{W} + \delta\bar{\psi}^z) \psi^z \end{aligned} \tag{2.4}$$

   The variation above vanishes for

$$\delta\psi^z = \epsilon^z \overline{W(z)}, \quad \delta\bar{\psi}^z = -\bar{\epsilon}^z \overline{W(z)} \tag{2.5}$$

   Similar analysis for the $\epsilon^{\bar{z}}$- and $\bar{\epsilon}^{\bar{z}}$-transformations lead to

$$\delta\psi^{\bar{z}} = \epsilon^{\bar{z}} W(z), \quad \delta\bar{\psi}^{\bar{z}} = -\bar{\epsilon}^{\bar{z}} W(z). \tag{2.6}$$

2. (*10 points*) Show that the integration measure $d^4\psi dz d\bar{z}$ also invariant under the same symmetries (to linear order is symmetry parameters)

Solution: Under the change of variables the integration measure $dzd\bar{z}d^4\psi$ transforms by super-determinant

$$dz'd\bar{z}'d^4\psi' = \text{sdet}(J) \cdot dzd\bar{z}d^4\psi \tag{2.7}$$

with J being Jacobian for the change of variables

$$\begin{aligned} z' &= z + \epsilon^z\bar{\psi}^z + \bar{\epsilon}^z\psi^z \\ \bar{z}' &= \bar{z} + \epsilon^{\bar{z}}\bar{\psi}^{\bar{z}} + \bar{\epsilon}^{\bar{z}}\psi^{\bar{z}} \\ \psi'_z &= \psi_z + \epsilon^z\overline{W(z)}, \\ \bar{\psi}'_z &= \bar{\psi}_z - \bar{\epsilon}^z\overline{W(z)} \\ \psi'_{\bar{z}} &= \psi_{\bar{z}} + \epsilon^{\bar{z}}W(z) \\ \bar{\psi}'_{\bar{z}} &= \bar{\psi}_{\bar{z}} - \bar{\epsilon}^{\bar{z}}W(z) \end{aligned} \tag{2.8}$$

In matrix form the Jacobian

$$J = \begin{pmatrix} 1 & 0 & -\bar{\epsilon}^z & -\epsilon^z & 0 & 0 \\ 0 & 1 & 0 & 0 & -\bar{\epsilon}^{\bar{z}} & -\epsilon^{\bar{z}} \\ 0 & \overline{\partial_z W}\epsilon^z & 1 & 0 & 0 & 0 \\ 0 & -\overline{\partial_z W}\bar{\epsilon}^z & 0 & 1 & 0 & 0 \\ \partial_z W\epsilon^{\bar{z}} & 0 & 0 & 0 & 1 & 0 \\ -\partial_z W\bar{\epsilon}^{\bar{z}} & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.9}$$

while the superdeterminant

$$\text{sdet}(J) = \text{Ber}(J) = 1 + \mathcal{O}(\epsilon^4). \tag{2.10}$$

3. (*10 points*) An integral of symmetry-invariant action with respect to symmetry-invariant measure localizes. What are the localization points for integral (2.2)?

Solution: The localization points are the the points where transformation of the odd variables vanishes

$$\begin{aligned} \delta\psi^z &= \epsilon^z\overline{W(z)} = 0 \\ \delta\bar{\psi}^z &= -\bar{\epsilon}^z\overline{W(z)} = 0 \\ \delta\psi^{\bar{z}} &= \epsilon^{\bar{z}}W(z) = 0, \\ \delta\bar{\psi}^{\bar{z}} &= -\bar{\epsilon}^{\bar{z}}W(z) = 0 \end{aligned} \tag{2.11}$$

which are roots of the polynomial $W(z)$.

4. (*10 points*) Evaluate the (Gaussian) integrals around the localization points.

Solution: Let us expand the polynomial $W(z)$ around one of it's zeroes z_0

$$W(z) = W(z_0) + (z - z_0)\partial W(z_0) + \frac{1}{2}(z - z_0)^2 \partial_z^2 W(z_0) + \mathcal{O}(z - z_0)^3 \tag{2.12}$$

Let us introduce a notation

$$\alpha = \partial W(z_0) \tag{2.13}$$

so the action near the localization point z_0 takes the form

$$S(z, \psi) = |\alpha|^2 |z - z_0|^2 - \alpha \psi^z \bar{\psi}^z - \bar{\alpha} \psi^{\bar{z}} \bar{\psi}^{\bar{z}} + ... \tag{2.14}$$

The partition function integral around the localization point becomes Gaussian integral

$$\begin{aligned} Z_W &= \frac{1}{2\pi i} \sum_{z_0 : W(z_0) = 0} \int_{\mathbb{C}} d^4\psi dz d\bar{z} \ \ e^{-|\alpha|^2 |z - z_0|^2 + \alpha \psi^z \bar{\psi}^z + \bar{\alpha} \psi^{\bar{z}} \bar{\psi}^{\bar{z}}} \\ &= \frac{1}{2\pi i} \sum_{z_0 : W(z_0) = 0} |\alpha|^2 \int_{\mathbb{C}} dz d\bar{z} \ \ e^{-|\alpha|^2 |z - z_0|^2} \\ &= \frac{1}{2\pi i} \sum_{z_0 : W(z_0) = 0} |\alpha|^2 \int_{\mathbb{R}^2} -2i dx dy \ \ e^{-|\alpha|^2 (x^2 + y^2)} \\ &= -\frac{1}{\pi} \sum_{z_0 : W(z_0) = 0} |\alpha|^2 \cdot \sqrt{\frac{\pi}{|\alpha|^2}} \sqrt{\frac{\pi}{|\alpha|^2}} \\ &= - \sum_{z_0 : W(z_0) = 0} 1 \end{aligned} \tag{2.15}$$

5. (*10 points*) The partition function Z_W is invariant under the shift $W(z) \to W(z) + tf(z)$ with $f(z)$ being holomorphic polynomial with complex coefficients of degree less then degree of W. We can use such invariance to deform the W of degree n to the standard form

$$W(z) = az^n, \ \ a \in \mathbb{C} \setminus 0 \tag{2.16}$$

Assuming $W(z)$ has form (2.16) evaluate the integral

$$Z_W = \frac{1}{2\pi i} \int_{\mathbb{C}} dz d\bar{z} \, e^{-W(z)\overline{W(z)}} \partial_z W(z) \overline{\partial_z W(z)} \tag{2.17}$$

Hint: It might be helpful to use radial coordinates r, ϕ on $\mathbb{C}$

$$z = re^{i\phi}, \quad \bar{z} = re^{-i\phi}, \quad r \in [0, +\infty), \quad \phi \in [0, 2\pi). \tag{2.18}$$

Solution: The integration measure

$$dzd\bar{z} = -2i\, rdr\, d\phi \tag{2.19}$$

while the integral in radial coordinates

$$\begin{aligned} Z_W &= \frac{1}{2\pi i}\int_{\mathbb{C}} dzd\bar{z}\; e^{-W(z)\overline{W(z)}}\partial_z W(z)\overline{\partial_z W(z)} \\ &= \frac{1}{2\pi i}\int_{\mathbb{C}} dzd\bar{z}\; e^{-az^n\bar{a}\bar{z}^n} naz^{n-1}\; \bar{a}n\bar{z}^{n-1} \\ &= \frac{n^2|a|^2}{2\pi i}\int_{\mathbb{C}} dzd\bar{z}\; e^{-|a|^2|z|^{2n}}|z|^{2n-2} = -\frac{n^2|a|^2}{\pi}\int_0^{2\pi} d\phi \int_0^\infty rdre^{-|a|^2r^{2n}}r^{2n-2} \\ &= -2n^2|a|^2\int_0^\infty e^{-|a|^2r^{2n}}\frac{1}{2n}d(r^{2n}) = -n \end{aligned} \tag{2.20}$$

We can compare our answer with the localization answer to arrive into relation

$$Z_W = -n = -\deg(W) = -\sum_{z_0 : W(z_0)=0} 1 \tag{2.21}$$

also known as the fundamental theorem of algebra!