Supersymmetric Quantum Mechanics and Morse Theory

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Solution to Homework 1: Simplicial models

1 Homology

Describe the chain complex, evaluate cohomology and Euler characteristic for the following manifolds

1. (*10 points*) Disc D^2, using single 2d simplex as model

   Solution:The simplicial model for a two dimensional disc D^2 is a single 2d simplex

$$\Delta_D = \Delta^2 = e_{012}. \tag{1.1}$$

There is a single 2d simplex so

$$C_2 = \mathbb{R}\langle e_{012}\rangle = \mathbb{R} \tag{1.2}$$

There are three 1d simplexes e_{12}, e_{02}, e_{01} and three 1d simplexes e_0, e_1 and e_2, so the corresponding vector spaces

$$C_0 = \mathbb{R}\langle e_0, e_1, e_2\rangle = \mathbb{R}^3, \;\; C_1 = \mathbb{R}\langle e_{12}, e_{02}, e_{01}\rangle = \mathbb{R}^3. \tag{1.3}$$

The boundary operation acts on C_1 in the following way

$$\partial(c_{ij}e_{ij}) = c_{ij}\partial e_{ij} = c_{ij}(e_j - e_i) = -c_{ij}e_i + c_{ij}e_j. \tag{1.4}$$

while the boundary operation for 2d simplex is

$$\partial(ce_{012}) = c(e_{12} - e_{02} + e_{01}) \tag{1.5}$$

The chain complex for Δ_D is of the form

$$C_\bullet(\Delta_D) = \mathbb{R} \xrightarrow{\partial_2} \mathbb{R}^3 \xrightarrow{\partial_1} \mathbb{R}^3 \longrightarrow 0 \tag{1.6}$$

$$c \longrightarrow (c, -c, c)$$

$$(c_{12}, c_{02}, c_{01}) \longrightarrow (-c_{02} - c_{01}, c_{01} - c_{12}, c_{02} + c_{12})$$

Let us evaluate homology

$$H_0(C_\bullet) = \frac{\ker \partial_0}{\mathrm{Im}\partial_1} = \mathbb{R}^3/\mathbb{R}^2 = \mathbb{R} \tag{1.7}$$

The image of ∂_1 is 2-dimensional since its kernel is 1-dimensional

$$\alpha(-c_{02} - c_{01}) + \beta(c_{01} - c_{12}) + \gamma(c_{02} + c_{12}) = 0 \Leftrightarrow \alpha = \beta = \gamma \in \mathbb{R} \tag{1.8}$$

$$H_1(C_\bullet) = \frac{\ker \partial_1}{\mathrm{Im}\partial_2} = \mathbb{R}/\mathbb{R} = 0 \tag{1.9}$$

$$H_2(C_\bullet) = \ker \partial_2 = 0 \tag{1.10}$$

The Euler characteristic is

$$\chi(D) = \chi(C_\bullet) = \sum (-1)^k \dim C_k = 3 - 3 + 1 = 1 \tag{1.11}$$

Summary

$$H_0^\Delta(D) = \mathbb{R}, \quad H_1^\Delta(D) = H_2^\Delta(D) = 0, \quad \chi(D) = 1. \tag{1.12}$$

2. (*10 points*) Disc D^2, using two 2d simplexes glued along one 1d-sub-simplex as a model

Solution: The simplicial model is the pair of simplexes

$$\Delta_{D^2} = \Delta^2 \cup \Delta^2 = e_{012} \cup e_{1'2'3}. \tag{1.13}$$

glued together along e_{12}

$$e_{1'2'} = e_{12}, \quad e_{1'} = e_1, \quad e_{2'} = e_2 \tag{1.14}$$

We have two 2d simplexes so

$$C_2 = \mathbb{R}\langle e_{012}, e_{1'2'3} \rangle = \mathbb{R}^2 \tag{1.15}$$

There are six 1d simplexes $e_{12}, e_{02}, e_{01}, e_{1'2'}, e_{1'3}, e_{2'3}$, but the gluing reduces this number to five. Let us use $e_{12}, e_{02}, e_{01}, e_{13}, e_{23}$ as basis for C_1 i.e.

$$C_1 = \mathbb{R}\langle e_{12}, e_{02}, e_{01}, e_{13}, e_{23} \rangle = \mathbb{R}^5 \tag{1.16}$$

There are six 0d simplexes $e_0, e_1, e_2, e_3, e_{1'}, e_{2'}$ with only 4 being independent. Let us use e_0, e_1, e_2, e_3 as a basis for C_0

$$C_0 = \mathbb{R}\langle e_0, e_1, e_2, e_3\rangle = \mathbb{R}^4 \tag{1.17}$$

The boundary operator on 1 simplexes is

$$\partial(c_{ij}e_{ij}) = c_{ij}\partial e_{ij} = c_{ij}(e_j - e_i) = -c_{ij}e_i + c_{ij}e_j. \tag{1.18}$$

The boundary operator on 2-simplexes is

$$\begin{aligned}\partial(ce_{012} + c'e_{1'2'3}) &= ce_{12} - ce_{02} + ce_{01} + c'e_{2'3} - c'e_{1'3} + c'e_{1'2'} \\ &= (c + c')e_{12} - ce_{02} + ce_{01} - c'e_{13} + c'e_{23}\end{aligned} \tag{1.19}$$

All our analysis can be summarized in the for of chain complex

$$\mathbb{R}^2 \xrightarrow{\partial_2} \mathbb{R}^5 \xrightarrow{\partial_1} \mathbb{R}^4 \longrightarrow 0$$

$$(c, c') \longrightarrow (c + c', -c, \;\; c, -c', \; c')$$

$$(\;\; c_{12} \;\;, c_{02}, c_{01}, c_{13}, c_{23}) \longrightarrow (-c_{02} - c_{01}, c_{01} - c_{12} - c_{13}, c_{02} + c_{12} - c_{23}, c_{13} + c_{23})$$

Let us evaluate homology

$$H_0(C_\bullet) = \frac{\ker \partial_0}{\mathrm{Im}\partial_1} = \mathbb{R}^4/\mathbb{R}^3 = \mathbb{R} \tag{1.20}$$

The image of ∂_1 is 3-dimensional since its kernel is 2-dimensional

$$H_1(C_\bullet) = \frac{\ker \partial_1}{\mathrm{Im}\partial_2} = \mathbb{R}^2/\mathbb{R}^2 = 0 \tag{1.21}$$

$$H_2(C_\bullet) = \ker \partial_2 = 0 \tag{1.22}$$

The Euler characteristic is

$$\chi(D) = \chi(C_\bullet) = \sum(-1)^k \dim C_k = 5 - 5 + 2 = 1 \tag{1.23}$$

Summary:

$$H_2^\Delta(D^2) = H_1^\Delta(D^2) = 0, \quad H_0^\Delta(D) = \mathbb{R}, \;\; \chi(D) = 1. \tag{1.24}$$

The answers for two simplex model of D^2 match with single simplex model.

3. (*10 points*) Cylinder $S^1 \times I$, using two 2d simplexes glued together along two 1d-sub-simplexes

Solution: The simplicial model is the pair of simplexes

$$\Delta_{S^1\times I} = \Delta^2 \cup \Delta^2 = e_{012} \cup e_{1'2'3}. \tag{1.25}$$

glued together according to

$$e_{1'2'} = e_{12}, \quad e_{1'3} = e_{02}, \quad e_{1'} = e_1 = e_0, \quad e_{2'} = e_2 = e_3 \tag{1.26}$$

We have two 2d simplexes so

$$C_2 = \mathbb{R}\langle e_{012}, e_{1'2'3}\rangle = \mathbb{R}^2 \tag{1.27}$$

There are six 1d simplexes $e_{12}, e_{02}, e_{01}, e_{1'2'}, e_{1'3}, e_{2'3}$, but the gluing reduces this number to four. Let us use $e_{12}, e_{02}, e_{01}, e_{23}$ as basis for C_1 i.e.

$$C_1 = \mathbb{R}\langle e_{12}, e_{02}, e_{01}, e_{23}\rangle = \mathbb{R}^4 \tag{1.28}$$

There are six 0d simplexes $e_0, e_1, e_2, e_3, e_{1'}, e_{2'}$ with only 2 being independent. Let us use e_0, e_2 as a basis for C_0

$$C_0 = \mathbb{R}\langle e_0, e_2\rangle = \mathbb{R}^2 \tag{1.29}$$

The boundary operator on 1 simplexes is

$$\begin{aligned}
&\partial(c_{12}e_{12} + c_{02}e_{02} + c_{01}e_{01} + c_{23}e_{23}) = c_{12}\partial e_{12} + c_{02}\partial e_{02} + c_{01}\partial e_{01} + c_{23}\partial e_{23} \\
&= c_{12}(e_2 - e_1) + c_{02}(e_2 - e_0) + c_{01}(e_1 - e_0) + c_{23}(e_3 - e_2) \\
&= c_{12}(e_2 - e_0) + c_{02}(e_2 - e_0) + c_{01}(e_0 - e_0) + c_{23}(e_2 - e_2) \\
&= (c_{12} + c_{02})e_2 - (c_{12} + c_{02})e_0
\end{aligned} \tag{1.30}$$

The boundary operator on 2-simplexes is

$$\begin{aligned}\partial(ce_{012}+c'e_{1'2'3}) &= ce_{12}-ce_{02}+ce_{01}+c'e_{2'3}-c'e_{1'3}+c'e_{1'2'} \\ &= ce_{12}-ce_{02}+ce_{01}+c'e_{2'3}-c'e_{02}+c'e_{12} \\ &= (c+c')e_{12}-(c+c')e_{02}+ce_{01}+c'e_{23}\end{aligned} \tag{1.31}$$

All our analysis can be summarized in the for of chain complex

$$\mathbb{R}^2 \xrightarrow{\partial_2} \mathbb{R}^4 \xrightarrow{\partial_1} \mathbb{R}^2 \longrightarrow 0$$

$$(c, c') \longrightarrow (c+c', -c-c', c, c')$$

$$(\ c_{12},\ \ c_{02},\ \ c_{01}, c_{23}) \longrightarrow (-c_{12}-c_{02}, c_{12}+c_{02})$$

Let us evaluate homology

$$H_0(C_\bullet) = \frac{\ker \partial_0}{\mathrm{Im}\partial_1} = \mathbb{R}^2/\mathbb{R}^1 = \mathbb{R} \tag{1.32}$$

The image of ∂_1 is 1-dimensional hence kernel is 3-dimensional

$$H_1(C_\bullet) = \frac{\ker \partial_1}{\mathrm{Im}\partial_2} = \mathbb{R}^3/\mathbb{R}^2 = \mathbb{R} \tag{1.33}$$

The image of ∂_2 is 2-dimensional since the kernel is 0-dimensional

$$H_2(C_\bullet) = \ker \partial_2 = 0 \tag{1.34}$$

The Euler characteristic is

$$\chi(S^1 \times I) = \chi(C_\bullet) = \sum (-1)^k \dim C_k = 2-4+2 = 0 \tag{1.35}$$

Summary

$$H_0^\Delta(S^1 \times I) = H_1^\Delta(S^1 \times I) = \mathbb{R}, \quad H_2^\Delta(S^1 \times I) = \mathbb{R}, \ \ \chi(S^1 \times I) = 0. \tag{1.36}$$

4. (*10 points*) Torus $T^2 = S^1 \times S^1$, using two 2d simplexes glued together

Solution: The simplicial model is the pair of simplexes

$$\Delta_{T^2} = \Delta^2 \cup \Delta^2 = e_{012} \cup e_{1'2'3}. \tag{1.37}$$

glued together according to

$$e_{1'2'} = e_{12}, \quad e_{1'3} = e_{02}, \quad e_{01} = e_{2'3}, \quad e_{1'} = e_1 = e_0 = e_{2'} = e_2 = e_3 \tag{1.38}$$

We have two 2d simplexes so

$$C_2 = \mathbb{R}\langle e_{012}, e_{1'2'3}\rangle = \mathbb{R}^2 \tag{1.39}$$

There are six 1d simplexes $e_{12}, e_{02}, e_{01}, e_{1'2'}, e_{1'3}, e_{2'3}$, but the gluing reduces this number to three. Let us use e_{12}, e_{02}, e_{01} as basis for C_1 i.e.

$$C_1 = \mathbb{R}\langle e_{12}, e_{02}, e_{01}\rangle = \mathbb{R}^3 \tag{1.40}$$

There are six 0d simplexes $e_0, e_1, e_2, e_3, e_{1'}, e_{2'}$ with only 1 is independent. Let us use e_0 as a basis for C_0

$$C_0 = \mathbb{R}\langle e_0\rangle = \mathbb{R} \tag{1.41}$$

The boundary operator on 1 simplexes is

$$\begin{aligned} &\partial(c_{12}e_{12} + c_{02}e_{02} + c_{01}e_{01}) = c_{12}\partial e_{12} + c_{02}\partial e_{02} + c_{01}\partial e_{01} \\ &= c_{12}(e_2 - e_1) + c_{02}(e_2 - e_0) + c_{01}(e_1 - e_0) \\ &= c_{12}(e_0 - e_0) + c_{02}(e_0 - e_0) + c_{01}(e_0 - e_0) = 0 \end{aligned} \tag{1.42}$$

The boundary operator on 2-simplexes is

$$\begin{aligned} \partial(ce_{012} + c'e_{1'2'3}) &= ce_{12} - ce_{02} + ce_{01} + c'e_{2'3} - c'e_{1'3} + c'e_{1'2'} \\ &= ce_{12} - ce_{02} + ce_{01} + c'e_{01} - c'e_{02} + c'e_{12} \\ &= (c + c')e_{12} - (c + c')e_{02} + (c + c')e_{01} \end{aligned} \tag{1.43}$$

All our analysis can be summarized in the for of chain complex

$$\mathbb{R}^2 \xrightarrow{\partial_2} \mathbb{R}^3 \xrightarrow{\partial_1} \mathbb{R}^1 \longrightarrow 0$$

$$(c, c') \longrightarrow (c + c', -c - c', c + c')$$

$$(\ c_{12},\ \ c_{02},\ \ c_{01}) \longrightarrow 0$$

Let us evaluate homology

$$H_0(C_\bullet) = \frac{\ker \partial_0}{\mathrm{Im} \partial_1} = \mathbb{R} \tag{1.44}$$

The image of ∂_1 is 0-dimensional hence kernel is 3-dimensional

$$H_1(C_\bullet) = \frac{\ker \partial_1}{\mathrm{Im} \partial_2} = \mathbb{R}^3/\mathbb{R}^1 = \mathbb{R}^2 \tag{1.45}$$

The image of ∂_2 is 1-dimensional since the kernel is 1-dimensional

$$H_2(C_\bullet) = \ker \partial_2 = \mathbb{R} \tag{1.46}$$

The Euler characteristic is

$$\chi(T^2) = \chi(C_\bullet) = \sum (-1)^k \dim C_k = 2 - 3 + 1 = 0 \tag{1.47}$$

Summary

$$H_0^\Delta(T^2) = H_2^\Delta(T^2) = \mathbb{R}, \quad H_1^\Delta(T^2) = \mathbb{R}^2, \ \ \chi(T^2) = 0. \tag{1.48}$$

5. (10 points) Sphere S^2, using two 2d simplexes glued together

Solution: The simplicial model is the pair of simplexes

$$\Delta_{S^2} = \Delta^2 \cup \Delta^2 = e_{012} \cup e_{1'2'3}. \tag{1.49}$$

glued together according to

$$\begin{gathered} e_{1'2'} = e_{12}, \ \ e_{1'3} = e_{10} = -e_{01}, \ \ e_{2'3} = e_{20} = -e_{02}, \\ e_{1'} = e_1, e_2 = e_{2'}, e_0 = e_3 \end{gathered} \tag{1.50}$$

We have two 2d simplexes so

$$C_2 = \mathbb{R}\langle e_{012}, e_{1'2'3}\rangle = \mathbb{R}^2 \tag{1.51}$$

There are six 1d simplexes $e_{12}, e_{02}, e_{01}, e_{1'2'}, e_{1'3}, e_{2'3}$, but the gluing reduces this number to three. Let us use e_{12}, e_{02}, e_{01} as basis for C_1 i.e.

$$C_1 = \mathbb{R}\langle e_{12}, e_{02}, e_{01}\rangle = \mathbb{R}^3 \tag{1.52}$$

There are six 0d simplexes $e_0, e_1, e_2, e_3, e_{1'}, e_{2'}$ with only 3 are independent. Let us use e_0, e_1, e_2 as a basis for C_0

$$C_0 = \mathbb{R}\langle e_0, e_1, e_2\rangle = \mathbb{R}^3 \tag{1.53}$$

The boundary operator on 1 simplexes is

$$\begin{aligned}
&\partial(c_{12}e_{12} + c_{02}e_{02} + c_{01}e_{01}) = c_{12}\partial e_{12} + c_{02}\partial e_{02} + c_{01}\partial e_{01} \\
&= c_{12}(e_2 - e_1) + c_{02}(e_2 - e_0) + c_{01}(e_1 - e_0) \\
&= (-c_{01} - c_{02})e_0 + (c_{01} - c_{12})e_1 + (c_{12} + c_{02})e_2
\end{aligned} \tag{1.54}$$

The boundary operator on 2-simplexes is

$$\begin{aligned}
\partial(ce_{012} + c'e_{1'2'3}) &= ce_{12} - ce_{02} + ce_{01} + c'e_{2'3} - c'e_{1'3} + c'e_{1'2'} \\
&= ce_{12} - ce_{02} + ce_{01} - c'e_{02} + c'e_{01} + c'e_{12} \\
&= (c + c')e_{12} - (c + c')e_{02} + (c + c')e_{01}
\end{aligned} \tag{1.55}$$

All our analysis can be summarized in the for of chain complex

$$\mathbb{R}^2 \xrightarrow{\partial_2} \mathbb{R}^3 \xrightarrow{\partial_1} \mathbb{R}^3 \longrightarrow 0$$

$$(c, c') \longrightarrow (c + c', -c - c', c + c')$$

$$(\ c_{12},\ \ c_{02},\ \ c_{01}) \longrightarrow (-c_{01} - c_{02}, c_{01} - c_{12}, c_{12} + c_{02})$$

Let us evaluate homology

$$H_0(C_\bullet) = \frac{\ker \partial_0}{\mathrm{Im}\partial_1} = \mathbb{R}^3/\mathbb{R}^2 = \mathbb{R} \tag{1.56}$$

The image of ∂_1 is 2-dimensional hence kernel is 1-dimensional

$$H_1(C_\bullet) = \frac{\ker \partial_1}{\mathrm{Im}\partial_2} = \mathbb{R}^1/\mathbb{R}^1 = 0 \tag{1.57}$$

The image of ∂_2 is 1-dimensional since the kernel is 1-dimensional

$$H_2(C_\bullet) = \ker \partial_2 = \mathbb{R} \tag{1.58}$$

The Euler characteristic is

$$\chi(T^2) = \chi(C_\bullet) = \sum (-1)^k \dim C_k = 2 - 3 + 3 = 2 \tag{1.59}$$

Summary

$$H_0^\Delta(S^2) = H_2^\Delta(S^2) = \mathbb{R}, \quad H_1^\Delta(S^2) = 0, \;\; \chi(S^2) = 2. \tag{1.60}$$

6. (*15 points*) Möbius strip

Solution: The simplicial model is the pair of simplexes

$$\Delta_M = \Delta^2 \cup \Delta^2 = e_{012} \cup e_{1'2'3}. \tag{1.61}$$

glued together according to

$$e_{1'2'} = e_{12}, \;\; e_{1'3} = e_{20} = -e_{02}, \;\; e_{1'} = e_1 = e_2 = e_{2'}, \; e_3 = e_0 \tag{1.62}$$

We have two 2d simplexes so

$$C_2 = \mathbb{R}\langle e_{012}, e_{1'2'3}\rangle = \mathbb{R}^2 \tag{1.63}$$

There are six 1d simplexes $e_{12}, e_{02}, e_{01}, e_{1'2'}, e_{1'3}, e_{2'3}$, but the gluing reduces this number to four. Let us use $e_{12}, e_{02}, e_{01}, e_{23}$ as basis for C_1 i.e.

$$C_1 = \mathbb{R}\langle e_{12}, e_{02}, e_{01}, e_{23}\rangle = \mathbb{R}^4 \tag{1.64}$$

There are six 0d simplexes $e_0, e_1, e_2, e_3, e_{1'}, e_{2'}$ with only 2 being independent. Let us use e_0, e_1 as a basis for C_0

$$C_0 = \mathbb{R}\langle e_0, e_1\rangle = \mathbb{R}^2 \tag{1.65}$$

The boundary operator on 1 simplexes is

$$\begin{aligned}
&\partial(c_{12}e_{12} + c_{02}e_{02} + c_{01}e_{01} + c_{23}e_{23}) = c_{12}\partial e_{12} + c_{02}\partial e_{02} + c_{01}\partial e_{01} + c_{23}\partial e_{23} \\
&= c_{12}(e_2 - e_1) + c_{02}(e_2 - e_0) + c_{01}(e_1 - e_0) + c_{23}(e_3 - e_2) \\
&= c_{12}(e_1 - e_1) + c_{02}(e_1 - e_0) + c_{01}(e_1 - e_0) + c_{23}(e_0 - e_1) \\
&= (c_{02} + c_{01} - c_{23})e_1 + (c_{23} - c_{01} - c_{02})e_0
\end{aligned} \tag{1.66}$$

The boundary operator on 2-simplexes is

$$\begin{aligned}
\partial(ce_{012} + c'e_{1'2'3}) &= ce_{12} - ce_{02} + ce_{01} + c'e_{2'3} - c'e_{1'3} + c'e_{1'2'} \\
&= ce_{12} - ce_{02} + ce_{01} + c'e_{2'3} + c'e_{02} + c'e_{12} \\
&= (c + c')e_{12} + (c' - c)e_{02} + ce_{01} + c'e_{23}
\end{aligned} \tag{1.67}$$

All our analysis can be summarized in the for of chain complex

$$\mathbb{R}^2 \xrightarrow{\partial_2} \mathbb{R}^4 \xrightarrow{\partial_1} \mathbb{R}^2 \longrightarrow 0$$

$$(c, c') \longrightarrow (c + c', c' - c, c, c')$$

$$(\ c_{12},\ \ c_{02},\ \ c_{01}, c_{23}) \longrightarrow (c_{23} - c_{01} - c_{02}, c_{02} + c_{01} - c_{23})$$

Let us evaluate homology

$$H_0(C_\bullet) = \frac{\ker \partial_0}{\mathrm{Im}\partial_1} = \mathbb{R}^2/\mathbb{R}^1 = \mathbb{R} \tag{1.68}$$

The image of ∂_1 is 1-dimensional hence kernel is 3-dimensional

$$H_1(C_\bullet) = \frac{\ker \partial_1}{\mathrm{Im}\partial_2} = \mathbb{R}^3/\mathbb{R}^2 = \mathbb{R} \tag{1.69}$$

The image of ∂_2 is 2-dimensional since the kernel is 0-dimensional

$$H_2(C_\bullet) = \ker \partial_2 = 0 \tag{1.70}$$

The Euler characteristic is

$$\chi(M) = \chi(C_\bullet) = \sum (-1)^k \dim C_k = 2 - 4 + 2 = 0 \tag{1.71}$$

Summary

$$H_0^{\Delta}(M) = H_1^{\Delta}(M) = \mathbb{R}, \quad H_2^{\Delta}(M) = \mathbb{R}, \;\; \chi(M) = 0. \tag{1.72}$$

The real valued homology are identical for Möbius strip and cylinder $S^1 \times I$.

7. (*15 points*) Klein bottle

Solution: The simplicial model is the pair of simplexes

$$\Delta_{T^2} = \Delta^2 \cup \Delta^2 = e_{012} \cup e_{1'2'3}. \tag{1.73}$$

glued together according to

$$e_{1'2'} = e_{12}, \;\; e_{1'3} = -e_{02}, \;\; e_{01} = e_{2'3}, \;\; e_{1'} = e_1 = e_0 = e_{2'} = e_2 = e_3 \tag{1.74}$$

We have two 2d simplexes so

$$C_2 = \mathbb{R}\langle e_{012}, e_{1'2'3}\rangle = \mathbb{R}^2 \tag{1.75}$$

There are six 1d simplexes $e_{12}, e_{02}, e_{01}, e_{1'2'}, e_{1'3}, e_{2'3}$, but the gluing reduces this number to three. Let us use e_{12}, e_{02}, e_{01} as basis for C_1 i.e.

$$C_1 = \mathbb{R}\langle e_{12}, e_{02}, e_{01}\rangle = \mathbb{R}^3 \tag{1.76}$$

There are six 0d simplexes $e_0, e_1, e_2, e_3, e_{1'}, e_{2'}$ with only 1 is independent. Let us use e_0 as a basis for C_0

$$C_0 = \mathbb{R}\langle e_0\rangle = \mathbb{R} \tag{1.77}$$

The boundary operator on 1 simplexes is

$$\begin{aligned} &\partial(c_{12}e_{12} + c_{02}e_{02} + c_{01}e_{01}) = c_{12}\partial e_{12} + c_{02}\partial e_{02} + c_{01}\partial e_{01} \\ &= c_{12}(e_2 - e_1) + c_{02}(e_2 - e_0) + c_{01}(e_1 - e_0) \\ &= c_{12}(e_0 - e_0) + c_{02}(e_0 - e_0) + c_{01}(e_0 - e_0) = 0 \end{aligned} \tag{1.78}$$

The boundary operator on 2-simplexes is

$$\begin{aligned}\partial(ce_{012} + c'e_{1'2'3}) &= ce_{12} - ce_{02} + ce_{01} + c'e_{2'3} - c'e_{1'3} + c'e_{1'2'} \\ &= ce_{12} - ce_{02} + ce_{01} + c'e_{01} + c'e_{02} + c'e_{12} \\ &= (c+c')e_{12} + (c'-c)e_{02} + (c+c')e_{01}\end{aligned} \tag{1.79}$$

All our analysis can be summarized in the for of chain complex

$$\mathbb{R}^2 \xrightarrow{\partial_2} \mathbb{R}^3 \xrightarrow{\partial_1} \mathbb{R}^1 \longrightarrow 0$$

$$(c, c') \longrightarrow (c+c', c'-c, c+c')$$

$$(\ c_{12},\ \ c_{02},\ \ c_{01}) \longrightarrow 0$$

Let us evaluate homology

$$H_0(C_\bullet) = \frac{\ker \partial_0}{\mathrm{Im}\partial_1} = \mathbb{R} \tag{1.80}$$

The image of ∂_1 is 0-dimensional hence kernel is 3-dimensional

$$H_1(C_\bullet) = \frac{\ker \partial_1}{\mathrm{Im}\partial_2} = \mathbb{R}^3/\mathbb{R}^2 = \mathbb{R} \tag{1.81}$$

The image of ∂_2 is 2-dimensional since the kernel is 0-dimensional

$$H_2(C_\bullet) = \ker \partial_2 = 0 \tag{1.82}$$

The Euler characteristic is

$$\chi(K) = \chi(C_\bullet) = \sum (-1)^k \dim C_k = 2 - 3 + 1 = 0 \tag{1.83}$$

Summary

$$H_0^\Delta(K) = H_1^\Delta(K) = \mathbb{R}, \quad H_2^\Delta(K) = 0, \ \ \chi(K) = 0. \tag{1.84}$$

2 Euler characteristic

Use simplicial model to describe the vector spaces C_k of the simplicial complex and evaluate the Euler characteristic for the following manifolds

1. (10 points) n-dimensional ball, using single n-dimensional simplex Δ^n as a model

Solution: The k-dimensional sub-simplexes are of the form

$$e_{i_1 \dots i_{k+1}}, \quad i_j \in \{0, ..., n\} \tag{2.1}$$

so we can use choice of $k+1$ elements form $\{0, ..., n\}$ to label them. Therefore the total number of k-sub-simplexes

$$\dim C_k = C_{n+1}^{k+1} = \frac{(n+1)!}{(k+1)!(n-k)!} \tag{2.2}$$

The Euler characteristic

$$\begin{aligned} \chi(B^n) = \chi(C_\bullet(\Delta^n)) &= \sum_{k=0}^{n} (-1)^k \dim C_k \\ &= \sum_{k=0}^{n} (-1)^k C_{n+1}^{k+1} = 1 - \left(1 + \sum_{k=0}^{n} (-1)^{k+1} \, 1^{n-k} \, C_{n+1}^{k+1} \right) \\ &= 1 + (1-1)^{n+1} = 1. \end{aligned} \tag{2.3}$$

2. (10 points) n-dimensional sphere, as a n-boundary of Δ^{n+1}

Solution: We can use the same logic as in previous part of the problem, while shifting $n \to n+1$ i.e.

$$\dim C_k = C_{n+2}^{k+1} = \frac{(n+2)!}{(k+1)!(n-k+1)!} \tag{2.4}$$

The simplicial complex for the model of n-sphere has C_n as the highest dimensional simplex The Euler characteristic

$$\begin{aligned} \chi(S^n) = \chi(C_\bullet) &= \sum_{k=0}^{n} (-1)^k \dim C_k \\ &= \sum_{k=0}^{n} (-1)^k C_{n+2}^{k+1} = 1 - \left(1 + \sum_{k=0}^{n} (-1)^{k+1} \, 1^{n+1-k} \, C_{n+2}^{k+1} + (-1)^{n+1} \right) - (-1)^{n+1} \\ &= 1 + (1-1)^{n+2} + (-1)^n = 1 + (-1)^n \end{aligned} \tag{2.5}$$

The quick check of the formula is look at S^1 and S^2, which we analyzed before

$$\chi(S^1) = 0 = 1 + (-1)^1, \quad \chi(S^2) = 2 = 1 + (-1)^2. \tag{2.6}$$