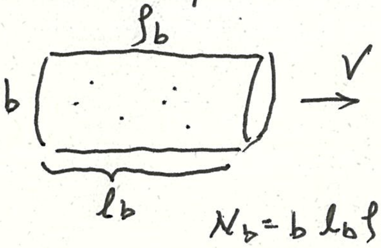
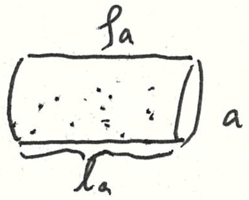
Lecture 1.

(12/.7/.18)

What can we learn about scattering from Poincare sym., crossing, unitarity? (without detail knowledge of Lagrangian).

- Collider experiment

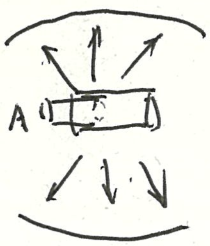




per cillinder:

100 billions protons

$N_b = b\, l_b \rho_b$



$\#$ of events of a given kind $= \sigma \dfrac{N_a N_b}{A}$

$\sigma \downarrow$ cross section

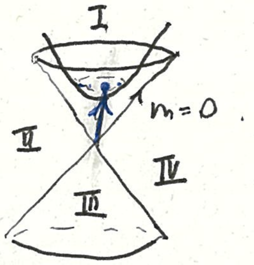
$[\sigma] = [A]$

unit : barn $10^{-24}\,cm^2$

Outgoing particles $\{P_1^\mu, P_2^\mu \cdots P_n^\mu\}$ $\quad \eta = [^{1}{}_{-1}{}_{\ddots}]$

$\{m_1 \quad m_2 \cdots m_n\}$

- Momentum space :



$E^2 - |\vec{P}_i|^2 = m^2$

Lorentz invariant: move I within I.

Volume : dLIPS

$$dV = \left[\prod_{i=1}^{n} d^4P_i \ \delta(P_i^2 - m_i^2)\ \theta(P_i^0)\right] \delta^4\Big(\sum_i P_i - (P_a + P_b)\Big)$$

θ: 1 ($P_i^0 > 0$), 0 (rest)

- Differential Cross Section

$$d\sigma_{P_aP_b \to 1 \cdots} = \frac{D}{2E_a 2E_b |v|}\, dLIPS.$$

$\uparrow$ Area $\qquad \uparrow$ relative velocity $\qquad \uparrow$ Lorentz inv.

$\longrightarrow$ Ex: without D, Check Lorentz transformation $\sim$ Area.

D : Lorentz invariant, contains of the information of the Dynamics.

· Hilbert space

Basis $\{|\psi_i\rangle\} \equiv B \qquad \{|\psi'_i\rangle\} \equiv B'$

For them to describe the same physical reality, Wigner 30's.

$\exists$ Operator $U \qquad |\psi\rangle \longrightarrow |\psi'\rangle = U|\psi\rangle$.

$$\langle|\psi\rangle = \sum_i \alpha_i |\psi_i\rangle, \quad P(\psi|\psi_i) = |\alpha_i|^2 = P(\psi'|\psi'_i)$$

$$= |\langle U\psi | U\psi_i\rangle|$$

(Adjoint operator θ: $\langle\varphi|\theta^\dagger|\psi\rangle = (\langle\psi|\theta|\varphi\rangle)^*$)

$$= |\langle\psi|U^\dagger U|\psi_i\rangle|$$

$\therefore U^\dagger U = \mathbb{1}$.

Scattering Physics Assumptions:

1) $\exists\ \mathcal{H}$ Hilbert space

2) The set of all possible incoming $\underbrace{\text{particles}}_{\text{states}}$ forms a basis (not event countable)

$$\{|0\rangle\} \oplus \{\underbrace{|p\rangle}_{\text{1 particle state}}\} \oplus \{|p_1 p_2\rangle\} \oplus \cdots \{|p_1 \cdots p_n\rangle\}$$

3) Same is true for outgoing, forms a basis

In our case: $|p_1 \cdots p_n\rangle_{out} = \sum_{m=0}^{\infty} \int d\,LIPS'_m \; C_{f,i} \; |p'_1 \cdots p'_m\rangle$.

$\exists\ \hat{S}$: scattering operator, $|p_1 \cdots p_n\rangle_{out} = \hat{S}\,|p_1 p_2 \cdots p_n\rangle_{in}$

$$\hat{S} - \mathbb{1} = \underset{\text{convention.}}{i}\,T$$

$$\langle p_a p_b | T | p_1 \cdots p_n\rangle = \mathcal{M}(p_a p_b \rightarrow p_1 \cdots p_n)\ \delta^4(p_a + p_b - \sum_i p_i).$$

istribution only has support on momenta conserved data.

invariant matrix element $\mathcal{M}$.

$$D = |\mathcal{M}(p_a p_b \rightarrow p_1 \cdots p_n)|^2.$$

What's the unitarity of $S^\dagger S = \mathbb{1} \Rightarrow T$?

$$(\mathbb{1} - iT^\dagger)(\mathbb{1} + iT) = \mathbb{1} - i(T^\dagger - T) + T^\dagger T \Rightarrow \mathbb{1}$$

$$\therefore \langle \text{Initial} | T^\dagger T | \text{Final} \rangle = i\langle I | T^\dagger | F \rangle - i\langle I | T | F \rangle$$

Insert resolution of identity

$$\mathbb{1} = |0\rangle\langle 0| + \int d^4p\, \theta(p_0)\, \delta(m^2 - p^2)\, |p\rangle\langle p| + \int dLIPS_2\, |p_1 p_2\rangle\langle p_1 p_2| + \cdots$$

$$\int d^4p\, \delta(p^2 - m^2)\theta(p_0) \underbrace{\langle I | T^\dagger | p\rangle}_{\langle p | T | I\rangle^*} \langle p | T | F\rangle + \cdots = i(\langle F | T | I\rangle)^* - i\langle I | T | F\rangle$$

$$M^*(p \to I)\, \delta^{(4)}(p_a + p_b - p) \qquad M(p \to F)\, \delta^{(4)}(p - \sum_i p_i)$$

$$\therefore \delta((p_a + p_b)^2 - m^2)\, \theta(p_0)\, M^*(p \to I)\, M(p \to F) + \cdots = i[M(F \to I)^* - M(I \to F)]$$

You need to fine tune initial momenta to make it non-vanishing.
here this first term is used to study the pole structure, to understand M.

- Assumption: without $\cdots$ term, (assume they don't contribute)
  $M(I \to F)$ is a rational function

In the momenta region close to $(p_a + p_b)^2 = m^2$,
because we have $\delta(x) = \lim_{\epsilon \to 0} \frac{1}{2\pi i}\left(\frac{1}{x - i\epsilon} - \frac{1}{x + i\epsilon}\right)$

$$\therefore \delta((p_a + p_b)^2 - m^2) = \frac{1}{2\pi i}\left(\frac{1}{(p_a+p_b)^2 - m^2 - i\epsilon} - \frac{1}{(p_a+p_b)^2 - m^2 + i\epsilon}\right)$$

$\left(\int dx\, \delta(x)\, f(x) = \int_C dz \cdots\right.$

$$\therefore \lim_{(p_a+p_b)^2 \to m^2} M(F \to I)^* = \frac{1}{2\pi} M^*(p \to I) \frac{1}{(p_a+p_b)^2 - m^2 - i\epsilon} M(p \to F)$$

$$\lim_{(p_a+p_b)^2 \to m^2} M(I \to F) = \frac{1}{2\pi} M^*(p \to I) \frac{1}{(p_a+p_b)^2 - m^2 + i\epsilon} M(p \to F).$$

Unitarity $\longrightarrow$ Tell us poles and residuals
(+ simplest assumption of rational function)

What is a particle?

Date

One - Particle State.

App. A & Chap 2 of Weinberg.

Poincare: $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}$. (Inhomogenious Lorentz).

Representation $U(\Lambda, a)$. unitary

- $U(\Lambda_1 a_1)\, U(\Lambda_2 a_2) = U(\Lambda_1 \Lambda_2,\ \Lambda_1 a_2 + a_1)$
- $\exists U^{-1}$: $U^{-1}(\Lambda, a)\, U(\Lambda, a) = U(1, 0) = \mathbb{1}$.

$\quad \therefore U(\Lambda^{-1}, -\Lambda^{-1} a)$

To study its algebra:

$$U(1, \epsilon) = \mathbb{1} - i \epsilon_{\mu} \underbrace{P^{\mu}}_{\text{Hermitian operator.}} \longleftarrow i\epsilon_{\mu}(P^{\mu\dagger} - P^{\mu}) \sim \mathcal{O}(\epsilon^2)$$

$$\begin{aligned} U(\Lambda, a)\, U(1, \epsilon)\, U^{\dagger}(\Lambda, a) &= U(\Lambda, a)\, U(1, \epsilon)\, U(\Lambda^{-1}, -\Lambda^{-1} a) \\ &= U(\Lambda, \Lambda\epsilon + a)\, U(\Lambda^{-1}, -\Lambda^{-1} a) \\ &= U(1, \Lambda\epsilon) \end{aligned}$$

$\therefore\ \epsilon_{\mu}\, U(\Lambda, a)\, P^{\mu}\, U^{\dagger}(\Lambda, a) = (\Lambda\epsilon)_{\mu} P^{\mu}$ ⟵ we want to know how P^{μ} transform

$(\Lambda\epsilon)_{\rho} P^{\rho} = (\Lambda_{\rho}{}^{\alpha}\, \epsilon_{\alpha}) P^{\rho}$

$\Lambda_{\rho}{}^{\alpha}$ — inverse Lorentz tran. $\Lambda_{\rho}{}^{\alpha} = \eta_{\rho m}\, \eta^{\alpha\beta}\, \Lambda^{m}{}_{\beta}$.

$\therefore\ U(\Lambda, a)\, P^{\mu}\, U^{-1}(\Lambda, a) = \Lambda_{\rho}{}^{\mu} P^{\rho}$ (*)

$$\epsilon_{\mu}(U P^{\mu} U^{\dagger}) = \underbrace{(\epsilon_{\mu} \Lambda_{\nu}{}^{\mu})}_{\Lambda\epsilon} P^{\nu}.$$

Definition of one particle state:

$P^{\mu} |p.\sigma\rangle = p^{\mu} |p.\sigma\rangle$ — σ: any other properties

$\underline{U(\Lambda.0)\, |p.\sigma\rangle = ?}$

from (*): $U(\Lambda^{-1}, 0)\, P^{\mu}\, U(\Lambda.0) = (\Lambda^{-1})_{\rho}{}^{\mu} P^{\rho}$

$\quad \therefore\ P^{\mu}\, U(\Lambda.0) = \Lambda^{\mu}{}_{\rho}\, U(\Lambda^{-1}) P^{\rho} = \Lambda^{\mu}{}_{\rho}\, U(\Lambda) P^{\rho}$

$\therefore\ P^{\mu}\, U(\Lambda) |p.\sigma\rangle = \Lambda^{\mu}{}_{\rho}\, U(\Lambda) P^{\rho} |p, \sigma\rangle = \Lambda^{\mu}{}_{\rho}\, p^{\rho} (U(\Lambda) |p.\sigma\rangle)$.

$\Rightarrow U(\Lambda) |p.\sigma\rangle \in \{ |\Lambda p, \sigma'\rangle \}\ \forall \sigma'$

$\therefore U(\Lambda)|p,\sigma\rangle = \sum_{\sigma'} \underbrace{C_{\sigma\sigma'}(\Lambda\cdot p)}_{?} |\Lambda p, \sigma'\rangle$

Classify all possible representation of the group.
(ways to get $C_{\sigma\sigma'}$)

- Trick: introduce reference vector k^μ, we can construct any other vector $p^\mu = \underbrace{\Lambda^\mu{}_\nu}_{L(p)} k^\nu$ (of course not unique).

Define $|p,\sigma\rangle = U(L(p))|k,\sigma\rangle$

$U(\Lambda)|p,\sigma\rangle = U(L(\Lambda p))\, U^{-1}(L(\Lambda p))\, U(\Lambda)\, U(L(p))\, |k,\sigma\rangle$

$= U(L(\Lambda p))\, U[\underbrace{L^{-1}(\Lambda p)\, \Lambda\, L(p)}_{W}]\, |k,\sigma\rangle$

$W \in \{$All possible Lorentz trans. that leaves k invariant.$\}$

$\downarrow$

Little group of k (subgroup of Lorentz). stablizer of k.

$U(W)|k,\sigma\rangle = \sum_{\sigma'} D_{\sigma'\sigma}(W)|k,\sigma'\rangle$

- Massive $m>0$ $k^2=m^2$ $k^\mu=(m,0,0,0)$ little group is $SO(3)$ rotation in space
  Irreps of $SO(3) \rightsquigarrow$ 5̶Ū̶(3) $\simeq SU(2)$ $\llcorner j = 0, \frac{1}{2}, 1, \frac{3}{2} \cdots$ spin.
- Massless: $ISO(2) \supset SO(2)$
  The inhomogeneous part: contineous spin

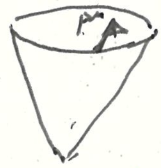
We have $U(\Lambda)|p,\sigma\rangle = U(L(\Lambda p))\left[\sum_{\sigma'} D_{\sigma\sigma'}(W)|k,\sigma'\rangle\right]$

$= \sum_{\sigma'} D_{\sigma\sigma'}(W)|\Lambda p,\sigma'\rangle$

for every irre rep of little group, we can induce a rep. of full Lorentz group

eg: Higgs spin 0, proton neutron not elementary but high spin (quarks <1% of mass)

Lecture 2 12/20/18



Massless (homogeous) little group: $SO(2)$

$U(\Lambda)|p,h\rangle = e^{ih\theta(\Lambda p)}|\Lambda p, h\rangle$ Invariant under 4π !!

Clever change of virables 80's "Spinor helicity formalism"

• $p^\mu \to \mathbb{P} = p^\mu \sigma_\mu$, $\det \mathbb{P} = 0$ (σ_μ : Paulis)

$$\downarrow$$

$$\underbrace{\begin{pmatrix}\alpha \\ \beta\end{pmatrix}}_{\lambda^p}(\gamma\ \ \delta) : \tilde{\lambda}^p \qquad \mathbb{P} = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix}$$

hence $p^\mu \to \{\rho\lambda^p, \rho^{-1}\tilde{\lambda}^p\}$ $\rho \in \mathbb{C}$ redundancy of rescaling.

Real $p^\mu \Leftrightarrow \mathbb{P}$ Hermition $\Leftrightarrow \lambda^p = \pm(\tilde{\lambda}^p)^*$

• $|p,h\rangle = |\lambda^p, \tilde{\lambda}^p, h\rangle$

Under ~~Lorentz~~ spin transform: $\mathbb{P}' = \Lambda \mathbb{P} \Lambda^\dagger$ $\Lambda \in SL(2,\mathbb{C})$

$$= \underbrace{\Lambda \lambda^p}_{e^{-\frac{i}{2}\theta(\Lambda)}\lambda^{\Lambda p}} \underbrace{(\tilde{\lambda}^p)^T \Lambda^\dagger}_{(\tilde{\lambda}^{\Lambda p})^T e^{\frac{i}{2}\theta(\Lambda)}}$$

denote $\mathbb{P}_{\alpha\dot{\alpha}} = \lambda^p_\alpha \tilde{\lambda}^p_{\dot{\alpha}}$ $\downarrow$ act as $\Lambda^\dagger$ from right

(spinor is fundamental rep. of $SL(2,\mathbb{C})$) Lorentz: $SL(2,\mathbb{C})/\mathbb{Z}_2$

• There are 2 $SL(2,\mathbb{C})$ invariants:

$$\lambda^p_\alpha \lambda^q_\beta \epsilon^{\alpha\beta} := \langle p\, q\rangle \left(= \det\begin{pmatrix}\lambda^p_1 & \lambda^q_1 \\ \lambda^p_2 & \lambda^q_2\end{pmatrix}\right)$$

$$\tilde{\lambda}^p_{\dot{\alpha}} \tilde{\lambda}^q_{\dot{\beta}} \epsilon^{\dot{\alpha}\dot{\beta}} := [p\, q]$$

$$\langle p\, q\rangle = -\langle q\, p\rangle \qquad [p\, q] = -[q\, p]$$

Coming back to our matrx element:

Test of Lorentz:

$\langle p_a p_b | U^\dagger(\Lambda) U(\Lambda) T U^\dagger(\Lambda) U(\Lambda) | p_1 \cdots p_n\rangle$.

$$\Rightarrow e^{\sum_{m=1}^{n} ih_m \theta(\Lambda p_m) - ih_a\theta(\Lambda p_a) - ih_b\theta(\Lambda p_b)} \langle \Lambda p_a\, \Lambda p_b | T | \Lambda p_1 \cdots \Lambda p_n\rangle$$

Our invariant matrix element:

$$\mathcal{M}(\{\lambda^a, \tilde\lambda^a, h_a\}, \{\lambda^b, \tilde\lambda^b, h_b\} \to \{\lambda^c, \tilde\lambda^c, h_c\}, \{\lambda^d, \tilde\lambda^d, h_d\})$$

= Rational function ($\langle ab\rangle$ $\langle ac\rangle$ .... $[ab]$, $[ac]$ ...)

(We want to build this)

rescaling: $\langle ab\rangle \to t_a t_b \langle ab\rangle$

$[ab] \to t_a^{-1} t_b^{-1} [ab]$

invariant under rescaling: $\langle ab\rangle [ab] = 2P_a \cdot P_b \; (= (P_a+P_b)^2 = S)$.

(how many ways can you prove this ?).

rescale the spinors in $\mathcal{M}$:

$$\mathcal{M}(\{t_a\lambda^a, t_a^{-1}\tilde\lambda^a, h_a\}, \{t_b\lambda^b, t_b^{-1}\tilde\lambda^b, h_b\} \to \cdots)$$

$$= \underline{t_a^{2h_a} t_b^{2h_b} t_c^{-2h_c} t_d^{-2h_d}} \; \mathcal{M}(\{\lambda^a, \tilde\lambda^a, h_a\}, \{\,^b\;\} \to \{\,^c\;\}, \{\,^d\;\})$$

$t_i = e^{i\theta(\Lambda P_i)}$

you can check this particle by particle ......

powers right $\to$ F will be fine under Lorentz.

Example:

3 particle: $P_1^\mu + P_2^\mu = P_3^\mu$

$P_1 \cdot P_2 = 0$ $\qquad$ $P_2 \cdot P_3 = 0$ $\qquad$ $P_1 \cdot P_3 = 0$

$\therefore \langle 1\,2\rangle [1\,2] = 0$ $\qquad$ $\langle 2\,3\rangle [2\,3] = 0$ $\qquad$ $\langle 1\,3\rangle [1\,3] = 0$.

Let's ... for a moment ... try complex momenta! (just to constrain rational function)

$\Downarrow$

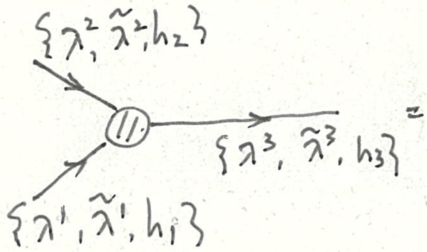
Freedom of spinor $\lambda \neq \pm \tilde\lambda^*$

we will have either

$\langle 1\,2\rangle = 0$ & $\langle 2\,3\rangle = 0$ & $\langle 1\,3\rangle = 0$

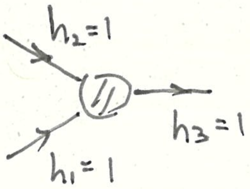
or

$[1\,2] = 0$ & $[2\,3] = 0$ & $[1\,3] = 0$.



$= F([12],[23],[31])$ or $F(\langle 12\rangle, \langle 23\rangle, \langle 31\rangle)$

rational function



$= F$? Requirement by rescaling: $F(t_1 t_2 \langle 12\rangle, t_2 t_3\langle 23\rangle, t_3 t_1 \langle 31\rangle)$

$$= t_1^2 t_2^2 \; t_3^{-2} \; F(\langle 12\rangle \langle 23\rangle \langle 31\rangle)$$

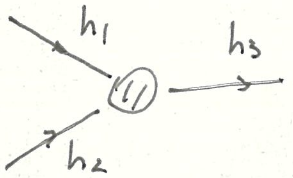
$$F = g \frac{\langle 12\rangle^3}{\langle 13\rangle\langle 32\rangle} \; ①$$

$[g]$ dimensionless.

which is actually the coupling constant!

$$\text{or}\left(\tilde g \frac{[13]\;[32]}{[12]^3} \; ②\right)$$

So in general:



$$= \sum_{x,y,z=-\infty}^{\infty} g_{xyz} \langle 12\rangle^x \langle 23\rangle^y \langle 31\rangle^z$$

simple:

$$\begin{cases} x+z = 2h_1 \\ x+y = 2h_2 \\ y+z = -2h_3 \end{cases} \Rightarrow \begin{cases} x = h_1+h_2+h_3 \\ y = -h_1+h_2-h_3 \\ z = h_1-h_2-h_3 \end{cases}$$

$$\text{so} = g \langle 12\rangle^x \langle 23\rangle^y \langle 31\rangle^z$$

$$\text{or} = \tilde g [12]^{-x} [23]^{-y} [31]^{-z}$$

Coming back to real momenta: ② is unphisical!

$$\langle \{P_1, 1\}, \{P_2, 1\} | T | \{P_3, 1\}\rangle = g \frac{\langle 12\rangle^3}{\langle 13\rangle \langle 32\rangle} \; \delta^4(P_1^\mu + P_2^\mu - P_3^\mu)$$

swith P_1, P_2, right side get minus

(bosons)

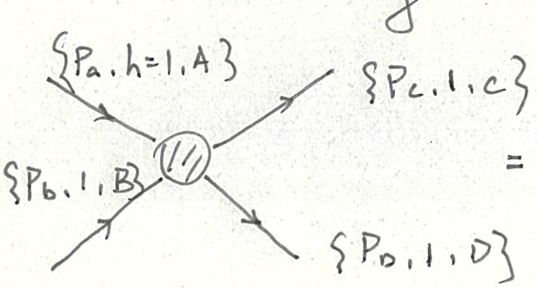
g has to be zero!

Hence: A single kind of helicity +1 particle can't interact (photons)

But with several species of +1 particle, it works.

$$\langle \{P_A, h_A=1\}, \{P_B, h_B=1\} | T | \{P_C, h_C=1\}\rangle = g_{AB}^{\;\;C} \frac{\langle AB\rangle^3}{\langle AC\rangle\langle CB\rangle} \qquad g_{AB}^{\;\;C} = -g_{BA}^{\;\;C}$$

2 → 2 Scattering



dimension m^{-2}

$$= \frac{\langle ab\rangle^2 [cd]^2}{s\,t\,u} (n_s\, s + n_t\, t + n_u\, u) \quad (*)$$

unitarity ⇒ only poles are s t u.

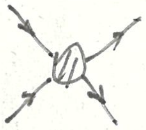
(Rescaling, has to transform as $t_a^{2} t_b^{2} t_c^{-2} t_d^{-2}$)

n_s, n_t, n_u don't depend on kinematics, but couplings

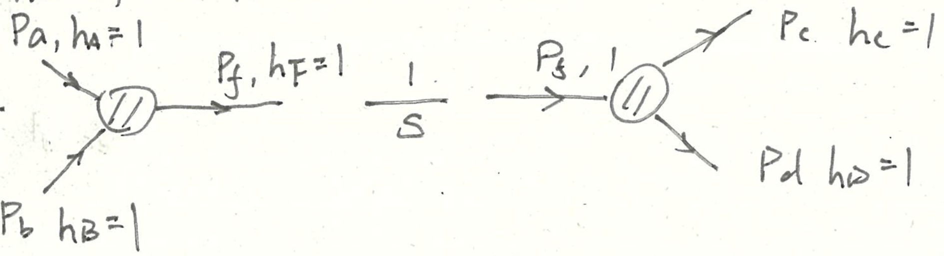
$$\lim_{s\to 0} (*) = \frac{\langle ab\rangle^2 [cd]^2}{t\,u} (n_t t - n_u t) \frac{1}{s} \qquad (s+t+u = m^2 = 0)$$

$$= \frac{\langle ab\rangle^2 [cd]^2}{\underbrace{\langle ab\rangle [ab]}_{u}} (n_t - n_u) \frac{1}{s}$$

On the other hand, we know



$= \sum_F$



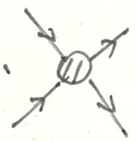
$$= \sum_F g_{AB}{}^F \frac{\langle a\, b\rangle^3}{\langle af\rangle \langle fb\rangle} \frac{1}{s} g_{CD}{}^F \frac{[c\, d]^3}{[cf][fd]}$$

look at $\langle af\rangle [fd] = \lambda^a_\alpha \underbrace{\lambda^f_\beta \tilde\lambda^f_{\dot\gamma}}_{\mathbb{P}^f_{\beta\dot\gamma} = \mathbb{P}^a_{\beta\dot\gamma} + \mathbb{P}^b_{\beta\dot\gamma}} \tilde\lambda^d_{\dot\rho}\, \epsilon^{\alpha\beta} \epsilon^{\dot\gamma\dot\rho}$

$$= \cancelto{0}{\langle a a\rangle} [ad] + \langle ab\rangle [bd]$$

$$[cf]\langle fb\rangle = [cd]\langle db\rangle$$

∴



$$= \sum_F g_{AB}{}^F g_{CD}{}^F \frac{\langle ab\rangle^2 [cd]^2}{\underbrace{\langle db\rangle [bd]}_{t=u}} \frac{1}{s} \Rightarrow n_t - n_u = \sum_F g_{AB}{}^F g_{CD}{}^F$$

Similarly for u, t channel

Add up s, t, u channels:

$$\sum_F \left(g^F_{AB}\, g^F_{CD} + g^F_{AC}\, g^F_{BD} + g^F_{AD}\, g^F_{BC} \right) = 0$$

Unitarity + Crossing + Lorentz $\Rightarrow$ Jacobi Identity.

$\Rightarrow$ It is governed by Lie algebras!

(and you need at least 3 species to make it work!

eg. $Z, W^{\pm}, \gamma$

$\uparrow$ Higgs.

$$\underbrace{\Lambda_1 \ \Lambda_2 \ \Lambda_3}_{SU(2)} \quad \underset{\underset{U(1)}{\uparrow}}{B}$$