Lemma 6.2
Let $\Omega_\theta \subset \mathbb{C}_t$ be a halfstrip centred around some ray $\mathbb{R}_\theta$. Suppose $\varphi \in \mathcal{E}^1(\Omega_\theta)$ is a holomorphic function on Ω_θ of exponential type at infinity. Then the Laplace transform of φ in the direction θ defines a holomorphic function

$$f(x) := \mathfrak{L}_\theta\big[\varphi\big](x) \tag{6.5}$$

on some Borel sector S whose opening $\mathsf{A} = (\theta - \pi/2, \theta + \pi/2)$ has angle π and bisecting direction θ. Furthermore, f is asymptotically smooth with factorial growth uniformly along A, and its leading-order term is 0. In symbols, $f \in \underline{\mathfrak{a}}^1(\mathsf{S})$. Furthermore, the formal Laplace transform $\widehat{f} := \widehat{\mathfrak{L}}[J\varphi]$ of the Taylor expansion of φ at 0 is the asymptotic expansion of f at 0: i.e., we have the following equality of formal power series:

$$\widehat{\mathfrak{L}}[J\varphi](x) = \widehat{f}(x) .$$

Proof. Our proof is similar to [LR16, Theorem 5.3.9]; see also [Mal95, p.182].

STEP 1: f IS WELL-DEFINED. We assume without loss of generality that $\theta = 0$, and denote Ω_θ simply by Ω. Let $K, L > 0$ be such that

$$\big|\varphi(t)\big| \leqslant K e^{L|t|} \qquad \forall t \in \Omega . \tag{6.6}$$

Fix any $R > L$. Then this exponential estimate implies that the Laplace integral

$$f(x) := \mathfrak{L}_+\big[\varphi\big](x) = \int_0^{+\infty} e^{-t/x}\varphi(t)\,\mathrm{d}t \tag{6.7}$$

is uniformly convergent for all x such that $\mathrm{Re}(1/x) > R$. Therefore, it defines a holomorphic function on the Borel sector $\mathsf{S} := \big\{x \;\big|\; \mathrm{Re}(1/x) > R\big\}$.

STEP 2: ASYMPTOTIC EXPANSION. Now we compute the asymptotic expansion of f by differentiating under the integral sign (thanks to the fact that the integral in (6.7) is uniformly convergent for all $x \in \mathsf{S}$) and using integration by parts:

$$\begin{aligned}
a_0 &:= \lim_{x\to 0} \int_0^{+\infty} e^{-t/x}\varphi(t)\,\mathrm{d}t = 0 \\
a_1 &:= \lim_{x\to 0} \partial_x \int_0^{+\infty} e^{-t/x}\varphi(t)\,\mathrm{d}t = \varphi(0) \\
a_2 &:= \lim_{x\to 0} \partial_x^2 \int_0^{+\infty} e^{-t/x}\varphi(t)\,\mathrm{d}t = \varphi'(0) \\
&\;\vdots \\
a_k &:= \lim_{x\to 0} \partial_x^k \int_0^{+\infty} e^{-t/x}\varphi(t)\,\mathrm{d}t = \varphi^{(k-1)}(0) .
\end{aligned}$$

Consulting the definition of the formal Laplace transform (5.22), we conclude immediately that $f \in \mathcal{A}(\mathsf{S})$; furthermore, $\widehat{f} = æ(f) = \widehat{\mathfrak{L}}\big[J\varphi\big]$ and $\widehat{\varphi}(t) = \widehat{\mathfrak{B}}\big[\widehat{f}\big]$ where

$$\widehat{\varphi}(t) := J\varphi(t) = \sum_{k=0}^{\infty} b_k t^k \quad \text{with} \quad b_k := \frac{\varphi^{(k)}}{k!} = \frac{a_{k+1}}{k!} .$$

STEP 3: ASYMPTOTIC BOUNDS. It remains to show that this function f satisfies the uniform factorial asymptotic bounds: that is, there are real constants $C, M > 0$ such that for all $x \in \mathsf{S}$ sufficiently small and all positive integers n,

$$\big|R_n(x)\big| \leqslant \left|f(x) - \sum_{k=0}^{n-1} a_k x^k\right| \leqslant CM^n n! |x|^n \ . \tag{6.8}$$

STEP 3.1: COVER S WITH TWO SMALLER SECTORS. To do so, we cover the sector S with two smaller sectors $\mathsf{S}_\pm$ whose openings $\mathsf{A}_\pm$ are strictly less than π; for example, we can take:

$$\mathsf{S}_\pm := \mathsf{S} \cap \widehat{\mathsf{S}}_\pm \quad \text{where} \quad \widehat{\mathsf{S}}_\pm := \Big\{ |x| < \tfrac{2}{R}, \quad \big|\arg(x) - \theta_\pm\big| < \tfrac{\pi}{3} \Big\} \quad \text{and} \quad \theta_\pm := \pm\tfrac{\pi}{4} \ .$$

The straight sectors $\widehat{\mathsf{S}}_\pm$ and the sectors $\mathsf{S}_\pm$ respectively have openings

$$\widehat{\mathsf{A}}_\pm := (\theta_\pm - \tfrac{\pi}{3}, \theta_\pm + \tfrac{\pi}{3}) \ , \quad \mathsf{A}_+ := (\theta_+ - \tfrac{\pi}{3}, +\tfrac{\pi}{2}) \ , \quad \mathsf{A}_- := (-\tfrac{\pi}{2}, \theta_- + \tfrac{\pi}{3}) \ .$$

The advantage of restricting to these subsectors $\mathsf{S}_\pm$ is that we get the following lower bound which will be used later in the proof:

$$x \in \mathsf{S}_\pm \quad \Longrightarrow \quad \mathrm{Re}(\omega_\pm / x) > c/|x| \qquad \text{where} \qquad \omega_\pm := e^{i\theta_\pm} = e^{\pm i\pi/4} \ , \tag{6.9}$$

and where $c := \sin(\frac{\pi}{3}) > 0$.

STEP 3.2: SHRINK Ω AND DEFORM INTEGRATION CONTOUR IN TWO WAYS. Let $\delta > 0$ be such that

$$\Omega = \big\{ \operatorname{dist}(t, \mathbb{R}_+) < 2\delta \big\}$$

and the formal Borel transform $\widehat{\varphi}$ is absolutely convergent in the disc $\Omega_0 = \{|t| < 2\delta\}$. Mark two points $t_\pm := \delta\omega_\pm \in \mathbb{C}_t$ and consider the following paths contained in Ω:

$$\gamma_\pm := [0, t_\pm] \qquad \text{and} \qquad \ell_\pm := t_\pm + \mathbb{R}_+ \ .$$

STEP 3.3: DECOMPOSE f ON EACH SUBSECTOR. Since the analytic continuation φ is holomorphic on Ω by assumption, we can decompose f on each subsector $\mathsf{S}_\pm$ as follows:

$$f(x) = f_\pm(x) + g_\pm(x)$$

where

$$f_\pm(x) := a_0 + \int_{\gamma_\pm} e^{-t/x} \varphi(t) \, \mathrm{d}t \qquad \text{and} \qquad g_\pm(t) := \int_{\ell_\pm} e^{-t/x} \varphi(t) \, \mathrm{d}t \ .$$

Claim 6.1. *Each function $f_\pm$ admits $\widehat{f}$ as its asymptotic expansion with factorial growth uniformly along $\mathsf{A}_\pm$: i.e., there is a constant $M_1 > 0$ (independent of the choice of $\pm$) such that, for all $n \geqslant 0$ and all $x \in \mathsf{S}_\pm$,*

$$\left| f_\pm(x) - \sum_{k=0}^{n-1} a_k x^k \right| \leqslant M_1^n n! |x|^n \ .$$

Claim 6.2. *Each function $g_\pm$ is asymptotic to 0 with factorial growth uniformly along $\mathsf{A}_\pm$: i.e., there is a constant $M_2 > 0$ (independent of the choice of $\pm$) such that, for all $n \geqslant 0$ and all $x \in \mathsf{S}_\pm$,*

$$\left|g_\pm(x)\right| \leqslant M_2^n n! |x|^n$$

The lemma now follows from these two claims by taking $M := \max\{M_1, M_2\}$. Now we prove these claims.

Proof of Claim 6.1. Each interval $\gamma_\pm$ is contained in the disc Ω_0 of absolute convergence of the power series $\widehat{\varphi}$, so $\varphi(t) = \widehat{\varphi}(t)$ for all $t \in \Omega_0$. Therefore, we are allowed to make the following computation:

$$\begin{aligned}
f_\pm(x) - \sum_{k=0}^{n-1} a_k x^k &= \int_{\gamma_\pm} e^{-t/x}\widehat{\varphi}(t)\,\mathrm{d}t - \sum_{k=1}^{n-1} a_k x^k &(6.10)\\
&= \int_0^{t_\pm} e^{-t/x}\sum_{k=0}^{\infty} b_k t^k\,\mathrm{d}t - \sum_{k=1}^{n-1}\frac{a_k}{(k-1)!}\int_0^{t_\pm\cdot\infty} t^{k-1}e^{-t/x}\,\mathrm{d}t\\
&= \sum_{k=0}^{\infty} b_k \int_0^{t_\pm} t^k e^{-t/x}\,\mathrm{d}t - \sum_{k=0}^{n-2}\frac{a_{k+1}}{k!}\int_0^{t_\pm\cdot\infty} t^k e^{-t/x}\,\mathrm{d}t\\
&= \sum_{k=0}^{\infty} b_k \int_0^{t_\pm} t^k e^{-t/x}\,\mathrm{d}t - \sum_{k=0}^{n-2} b_k \int_0^{t_\pm\cdot\infty} t^k e^{-t/x}\,\mathrm{d}t\\
&= \sum_{k=0}^{\infty} b_k \omega_\pm^{k+1}\int_0^{\delta} s^k e^{-s\omega_\pm/x}\,\mathrm{d}s - \sum_{k=0}^{n-2} b_k\omega_\pm^{k+1}\int_0^{+\infty} s^k e^{-s\omega_\pm/x}\,\mathrm{d}s\\
&= \sum_{k=n-1}^{\infty} b_k \omega_\pm^{k+1}\int_0^{\delta} s^k e^{-s\omega_\pm/x}\,\mathrm{d}s - \sum_{k=0}^{n-2} b_k\omega_\pm^{k+1}\int_\delta^{+\infty} s^k e^{-s\omega_\pm/x}\,\mathrm{d}s\ .
\end{aligned}$$

In the third line, we were allowed to interchange integration and summation because the series $\sum b_k t^k e^{-t/x}$ is absolutely convergent for all t in the interval $[0, t_\pm]$. (Indeed, from the inequality (6.9), $\operatorname{Re}(t/x) = |t|\operatorname{Re}(\omega_\pm/x) > 0$ because $x \in \mathsf{S}_\pm$, and so $\left|t^k e^{-t/x}\right| \leqslant |t|^k \leqslant |t_\pm|^k$ for all $t \in [0, t_\pm]$.) In the fifth line, we made the substitution $t = s\omega_\pm$ in both integrals.

The point of this computation is that the constraints on s and k in both the first and the second summation terms lead to the same bound on s^k:

$$\left.\begin{matrix} s \leqslant \delta & \text{and} & k \geqslant n-1 \\ s \geqslant \delta & \text{and} & k < n-1 \end{matrix}\right\} \implies \left(\frac{s}{\delta}\right)^{k-(n-1)} \leqslant 1 \ \Leftrightarrow\ s^k \leqslant s^{n-1}\delta^{k-n+1}\ .$$

So both integrals in the last line of (6.10) can be bounded above by the same quantity:

$$\left.\begin{matrix} \int_0^\delta s^k e^{-s\omega_\pm/x}\,\mathrm{d}s \\ \int_\delta^\infty s^k e^{-s\omega_\pm/x}\,\mathrm{d}s \end{matrix}\right\} \leqslant \delta^{k-n+1}\int_0^{+\infty} s^{n-1}e^{-sc/|x|}\,\mathrm{d}s \leqslant \delta^{k-n+1} n! \frac{|x|^n}{c^n}\ ,$$

where we again used the inequality (6.9). As a result, for all $x \in \mathsf{S}$, we obtain the following bound, valid for every $n \geqslant 1$:

$$\left|f_\pm(x) - \sum_{k=0}^{n-1} a_k x^k\right| \leqslant \sum_{k=0}^{\infty} |b_k|\delta^{k-n+1} n! \frac{|x|^n}{c^n} = |x|^n C_1 M_1^n n!\ ,$$

where $C_1 := \delta \sum_{k=0}^{\infty} |b_k| \delta^k$ and $M_1 := 1/c\delta$. Note that C_1 is finite because $\delta = |t_\pm|$ and $t_\pm$ is contained in the disc Ω_0 of uniform convergence of the power series $\widehat{\varphi}$. ☐

PROOF OF CLAIM 6.2. Parameterise the path $\ell_\pm$ as $t(s) = t_\pm + s$ for $s \in \mathbb{R}_+$. Then using the exponential estimate (6.6) it is easy to show that for all $x \in \mathsf{S}_\pm$, the function $g_\pm(x)$ is exponentially decaying:

$$\begin{aligned} \big|g_\pm(x)\big| &\leqslant \int_0^{+\infty} e^{-\operatorname{Re}(t_\pm/x)} e^{-s\operatorname{Re}(1/x)} \Big|\varphi\big(t_\pm + s\big)\Big| \, \mathrm{d}s \\ &\leqslant K e^{\delta L} e^{-c\delta/|x|} \int_0^{+\infty} e^{-s(R-L)} \\ &\leqslant C_2 e^{-M_2/|x|} \end{aligned}$$

where $C_2 := K e^{\delta L}(R-L)^{-1}$ and $M_2 := c\delta$. In particular, it follows that for every $n \in \mathbb{Z}_+$ we have the bound

$$\big|g_\pm(x) x^{-n}\big| \leqslant C_2 e^{-M_2/|x|} |x|^{-n} \ .$$

For $n = 0$, the claim is obviously true, so let us assume that $n \geqslant 1$ and analyse the real-valued function $r \mapsto r^{-n} e^{-M_2/r}$ on $\mathbb{R}_+$. It achieves its maximum value at $r = M_2/n$, so upon using Stirling's bounds, we obtain the desired estimate:

$$\big|g_\pm(x) x^{-n}\big| \leqslant C_2 K^{-n} n^n e^{-n} \leqslant C_2 M_2^n n! \ .$$ ☐

This completes the proof of Lemma 6.2. ■

Lemma 6.3
Let S be a sector with opening angle π and bisecting direction θ. Suppose $f \in \underline{\mathcal{A}}^1(\mathsf{S})$ is a holomorphic function on S which is uniformly asymptotically smooth with factorial growth. Then the Borel transform of f in the direction θ defines a holomorphic function

$$\varphi(t) := \mathfrak{B}_\theta\big[f\big](t) \tag{6.17}$$

of exponential type at infinity in some halfstrip Ω_θ centred around the ray $\mathbb{R}_\theta$. In symbols, $\varphi \in \mathcal{E}^1(\Omega_\theta)$. Furthermore, $f(x)$ can be expressed in terms of the Laplace transform of φ:

$$f = a_0 + \mathfrak{L}_\theta[\,\varphi\,] \ . \tag{6.18}$$

In particular, the formal Borel transform of the asymptotic expansion of f is the Taylor series expansion of φ at 0; that is, if $\widehat{f} := æ(f)$ and $\widehat{\varphi} := J\varphi \in \mathbb{C}\{t\}$, then there is a disc $\mathbb{D}$ around the origin in $\mathbb{C}_t$ such that for all $t \in \mathbb{D}$,

$$\widehat{\mathfrak{B}}[\widehat{f}](t) = J\varphi(t) \ . \tag{6.19}$$

Explicitly, formulas (6.17) and (6.18) read as follows:

$$f(x) = a_0 + \mathfrak{L}_\theta[\,\varphi\,](x) = a_0 + \int_{(\theta)} e^{-t/x}\varphi(t)\,\mathrm{d}t \ , \tag{6.20}$$

$$\varphi(t) = \mathfrak{B}_\theta[\,f\,](t) = \frac{1}{2\pi i}\oint_{(\theta)} e^{t/x} f(x)\frac{\mathrm{d}x}{x^2} \ . \tag{6.21}$$

Identities (6.20) and (6.21) are sometimes called the ***Borel-Laplace identities***.

Proof. Although the proof of this lemma may be long, the strategy is straightforward: we just need to verify directly that φ has all the desired properties. This verification is a combination of standard techniques from real and complex analysis which crucially rely on the fact that f admits asymptotics with factorial growth as $x \to 0$ uniformly along A.

STEP 1: SETUP. Let

$$\widehat{f}(x) := \sum_{k=0}^{\infty} a_k x^k \in \mathbb{C}^1[\![x]\!]$$

be the asymptotic expansion of f. We assume without loss of generality that $\theta = 0$ and S is the Borel sector

$$\mathsf{S} := \left\{ \operatorname{Re}(1/x) > \widehat{R} \right\}$$

with inverse-radius $\widehat{R} > 1$ so large that there are constants $C, M > 0$ such that for all $n \geqslant 0$, and all $x \in \mathsf{S}$ sufficiently small,

$$\big|R_n(x)\big| \leqslant \left| f(x) - \sum_{k=0}^{n-1} a_k x^k \right| \leqslant CM^n n! |x|^n . \tag{6.22}$$

By Lemma 4.2, these constants C, M can be chosen such that the coefficients a_k of the asymptotic expansion $\widehat{f}$, and therefore the coefficients b_k of the formal Borel transform $\widehat{\varphi} := \widehat{\mathfrak{B}}[\widehat{f}]$, satisfy the following bounds:

$$(\forall k \in \mathbb{Z}_{\geqslant 0}) \qquad |a_k| \leqslant CM^k k! \quad \text{and} \quad |b_k| \leqslant CM^k . \tag{6.23}$$

Then the power series $\widehat{\varphi}$ is absolutely convergent on the disc

$$\widehat{\mathbb{D}} := \left\{ t \in \mathbb{C} \;\middle|\; |t| < \widehat{r} \right\}$$

of radius $\widehat{r} := 1/M$.

STEP 2: COVER $\mathbb{R}_+$ WITH DISCS. Fix any $r \in (0, \widehat{r})$. For every point t_0 on the real axis $\mathbb{R}_t$ in the complex plane $\mathbb{C}_t$, consider the following interval and disc centred at t_0:

$$\mathbb{I}(t_0) := \left\{ t \in \mathbb{R} \;\middle|\; |t - t_0| < r \right\} \subset \mathbb{R}_t , \tag{6.24}$$

$$\mathbb{D}(t_0) := \left\{ t \in \mathbb{C} \;\middle|\; |t - t_0| < r \right\} \subset \mathbb{C}_t . \tag{6.25}$$

Obviously, $\mathbb{I}(t_0) = \mathbb{D}(t_0) \cap \mathbb{R}_t$. We also define $\widehat{\mathbb{I}} := \widehat{\mathbb{D}} \cap \mathbb{R}_t$. Let Ω be the strip neighbourhood of the positive real axis $\mathbb{R}_+$ of thickness r:

$$\Omega := \left\{ t \in \mathbb{C}_t \;\middle|\; \operatorname{dist}(t, \mathbb{R}_+) < r \right\} \subset \widehat{\mathbb{D}} \cup \bigcup_{t_0 > \widehat{r}} \mathbb{D}(t_0) . \tag{6.26}$$

STEP 3: FIVE CLAIMS. The rest of the proof can be broken down into a sequence of five claims. The first task is to check that formula (6.21) actually makes sense and defines a holomorphic function near $t = 0$ which coincides with the power series $\widehat{\varphi}$.

Claim 6.1. *The function φ given by (6.21) is well-defined for all $t \in \mathbb{R}_+$.*

So, in particular, φ is a function of the real variable t on the interval $\widehat{\mathbb{I}}$, where it can be compared with the power series $\widehat{\varphi}$.

Claim 6.2. *The power series $\widehat{\varphi}$ converges uniformly to φ for all $t \in \widehat{\mathbb{I}} = \widehat{\mathbb{D}} \cap \mathbb{R}_t$.*

Therefore, the convergent power series $\widehat{\varphi}$ is the analytic continuation of φ from the open interval $\widehat{\mathbb{I}}$ to the open disc $\widehat{\mathbb{D}}$. Next, we verify that φ is itself the analytic continuation of $\widehat{\varphi}$ along $\mathbb{R}_+$.

Claim 6.3. *The function φ is infinitely-differentiable at every point $t \in \mathbb{R}_+$. Furthermore, the constants $C, M > 0$ can be taken so large that all derivatives of φ satisfy the following exponential bound: for all $n \in \mathbb{Z}_{\geqslant 0}$, all $R > \widehat{R}$, and all $t \in \mathbb{R}_+$,*

$$\big|\partial_t^n \varphi(t)\big| \leqslant C M^n n! e^{Rt} \; . \tag{6.27}$$

Claim 6.4. *For every $t_0 \in \mathbb{R}_+$ with $t_0 > \widehat{r}$, the Taylor series $J_{t_0}\varphi$ of φ centred at the point t_0 is absolutely convergent to φ on the interval $\mathbb{I}(t_0)$.*

So the Taylor series $J_{t_0}\varphi$ is absolutely convergent on the whole disc $\mathbb{D}(t_0)$ and defines the analytic continuation of φ from the interval $\mathbb{I}(t_0)$ to the disc $\mathbb{D}(t_0)$. Since t_0 is arbitrary, this means that the function φ, defined by formula (6.21), admits analytic continuation to a holomorphic function (which we continue to denote by φ) on the tubular neighbourhood Ω. All that remains is to show that the Laplace transform of φ is well-defined and satisfies the equality (6.20).

Claim 6.5. *The holomorphic function φ on Ω has exponential type at infinity (i.e., φ satisfies the bound (6.6)), so its Laplace transform is well-defined and satisfies the equality (6.20) for all $t \in \mathbb{R}_+$.*

STEP 4: PROOFS OF THE FIVE CLAIMS. Now we prove these claims.

PROOF OF CLAIM 6.1. We look at the second remainder term

$$R_2(x) = f(x) - \big(a_0 + x a_1\big) \; .$$

If the Borel transforms $\mathfrak{B}[\,a_0\,], \mathfrak{B}[\,x a_1\,], \mathfrak{B}[\,R_2\,]$ are well-defined, it will follow that $\varphi(t)$ given by (6.21) is a well-defined function on $\mathbb{R}_+$. The first two are obviously well-defined, so we just need to examine $\mathfrak{B}[\,R_2\,]$. The asymptotic condition (6.22) with $n = 2$ reads $\big|R_2(x)\big| \leqslant 2! C M^2 |x|^2$. So if β is the Borel circle with any inverse-radius $R > \widehat{R}$, then the integral over β is well-defined because

$$\frac{1}{2\pi} \int_\beta \big|R_2(x)\big| e^{t\,\mathrm{Re}(1/x)} \left|\frac{\mathrm{d}x}{x^2}\right| \leqslant \frac{1}{\pi} C M^2 e^{Rt} \int_\beta |\,\mathrm{d}x\,| \leqslant C M^2 e^{Rt} R^{-1} \; .$$

Moreover, the integral only depends on the homotopy class (with fixed endpoints at $\pm\pi/2$) of the integration path β because because the integrand is holomorphic in the sector and the integral over any Borel circle β decays as the radius R increases. □

PROOF OF CLAIM 6.2. For any $n \geqslant 2$, take the identity

$$f(x) = \sum_{k=0}^{n-1} a_k x^k + R_n(x) \; ,$$

and plug it into formula (6.21) for φ to get

$$\varphi(t) = \sum_{k=0}^{n-2} a_k t^k + E_n(t) \quad \text{where} \quad E_n(t) := \frac{1}{2\pi i} \int_\gamma R_n(x) e^{t/x} \frac{\mathrm{d}x}{x^2} \ . \tag{6.28}$$

Thus, to prove this claim, we have to show that the error term $E_n(t)$ goes to zero as $n \to \infty$ for all $t \in \widehat{\mathbb{I}}$. For any $R > \widehat{R}$, let β_R be the Borel circle with radius R. Note that E_n is independent of R because of Claim 6.1. So for all $t > 0$, we find:

$$\big|E_n(t)\big| \leqslant \tfrac{1}{2\pi} \int_{\beta_R} \big|R_n(x)\big| e^{Rt} \left|\frac{\mathrm{d}x}{x^2}\right| \leqslant \tfrac{1}{2\pi} C M^n n! e^{Rt} \int_{\beta_R} \left| x^n \frac{\mathrm{d}x}{x^2} \right| \leqslant C M^n n! e^{Rt} R^{-n} \ .$$

Now, for any fixed t, look at the real-valued function $R \mapsto e^{Rt} R^{-n}$ defined for all $R > 0$. It achieves its minimum at $R = n/t$ with value $e^n n^{-n} t^n$. Fix a sufficiently large integer $N > 0$ such that $N/t_0 > R$. Then it follows that for all $n \geqslant N$,

$$\big|E_n(t)\big| \leqslant C M^n n! e^n n^{-n} t^n \leqslant C M^n e n^{1/2} t^n \ ,$$

where we used one of Stirling's bounds. Since $t < \widehat{r} = 1/M$, it follows that $|E_n(t)|$ goes to 0 as $n \to \infty$. □

Proof of Claim 6.3. Let $n \geqslant 0$ be any integer. We claim that the n-th derivative $\partial_t^n \varphi$ exists at every $t \in \mathbb{R}_+$. To do this, just like in (6.28), we look at the expression

$$\varphi(t) = \sum_{k=0}^{n} b_k t^k + E_{n+2}(t) \quad \text{where} \quad E_{n+2}(t) = \frac{1}{2\pi i} \int_\beta R_{n+2}(x) e^{t/x} \frac{\mathrm{d}x}{x^2} \ .$$

Then we just need to verify that the n-th derivative $\partial_t^n E_{n+2}$ exists at every $t \in \mathbb{R}_+$. In order to be able to swap differentiation and integration, we must first show that the n-th derivative of the integrand (which of course exists at every $t \in \mathbb{R}_+$) is bounded by an integrable function independent of t. For any $R > \widehat{R}$, restrict to the Borel circle γ with inverse-radius R. For all $t \in \mathbb{R}_+$, we find:

$$\left| \partial_t^n \left(R_{n+2}(x) \frac{e^{t/x}}{x^2} \right) \right| = \left| R_{n+2}(x) \frac{e^{t/x}}{x^{n+2}} \right| \leqslant C M^{n+2} (n+2)! e^{Rt} \ . \tag{6.29}$$

On any bounded interval in $\mathbb{R}_+$, the righthand side can be bounded by a constant independent of t (although it depends on the interval) which is of course integrable. Therefore, the derivative $\partial_t^n \varphi$ exists at every $t \in \mathbb{R}_+$ and equals

$$\partial_t^n \varphi(t) = n! b_n + \partial_t^n E_{n+2}(t) = n! b_n + \frac{1}{2\pi i} \int_\beta R_{n+2}(x) e^{t/x} \frac{\mathrm{d}x}{x^{n+2}} \ . \tag{6.30}$$

It remains to demonstrate the bounds (6.27). First, note that the integral $\int_\beta \left| x^n \frac{\mathrm{d}x}{x^2} \right|$ is convergent since $n \geqslant 2$. So we can combine (6.23), (6.29), and (6.30) to deduce:

$$\big|\partial_t^n \varphi(t)\big| \leqslant n! C M^n + C M^{n+2} (n+2)! e^{Rt} = C M^n n! \Big(e^{-Rt} + M^2 (n+2)(n+1) \Big) e^{Rt} \ .$$

Now, choose any number $c > 1$ and let $\tilde{M} := cM$. Then the above expression equals

$$C \tilde{M}^n n! \Big(c^{-n} e^{-Rt} + M^2 \tfrac{(n+2)(n+1)}{c^n} \Big) e^{Rt}$$

Since $e^{-Rt} < e^{-t\widehat{R}}$ and $c > 1$, the expression in the brackets is bounded by a constant $\tilde{c} > 1$ which is independent of n and R. Thus, if we let $\tilde{C} := \tilde{c}C$, we find that $\left|\partial_t^n \varphi(t)\right| \leqslant \tilde{C}\tilde{M}n!$. But since $C \leqslant \tilde{C}$ and $M \leqslant \tilde{M}$, we can redefine C, M to be the larger constants $\tilde{C}, \tilde{M}$. □

PROOF OF CLAIM 6.4. Fix any $R > \widehat{R}$ and any point $t_0 > \widehat{r}$ on the real line $\mathbb{R}_+$. We test the Taylor series

$$J_{t_0}\varphi(t) = \sum_{n=0}^{\infty} \frac{1}{n!}\partial_t^n \varphi(t_0)\big(t - t_0\big)^n$$

for absolute convergence on the interval $\mathbb{I}(t_0)$. Using the bound (6.27), we find:

$$\sum_{n=0}^{\infty} \frac{1}{n!}\left|\partial_t^n \varphi(t_0)\right| \cdot \left|t - t_0\right|^n \leqslant Ce^{Rt} \sum_{n=0}^{\infty} \tfrac{(n+2)!}{n!}(Mr)^n \leqslant C_0 \sum_{n=0}^{\infty} \tfrac{n^2+3n+2}{2^n} < \infty \;,$$

where $C_0 := Ce^{R(t_0+r)}$. To see that the Taylor series $J_{t_0}\varphi$ converges to φ on the interval $\mathbb{I}(t_0)$, we write the remainder in its mean-value form (a.k.a. Lagrange form):

$$G_m(t) := \varphi(t) - \sum_{n=0}^{m} \frac{1}{n!}\partial_t^n \varphi(t_0)(t - t_0)^n = \frac{1}{(m+1)!}\partial_t^{m+1}\varphi(t_*)\big(t - t_0\big)^{m+1}$$

for some point $t_* \in \mathbb{I}(t_0)$ that lies between t_0 and t. Using (6.27) again, and the fact that $r < 1/M$, this yields a bound that goes to 0 as $m \to \infty$:

$$\left|G_m(t)\right| \leqslant \frac{(m+3)!}{(m+1)!} C_0 (Mr)^m = C_0(m+3)(m+2)(Mr)^m \;.$$

□

PROOF OF CLAIM 6.5. Fix any $t \in \Omega$ and assume without loss of generality that $\mathrm{Re}(t) > \widehat{r}$. Then t is necessarily contained in a disc $\mathbb{D}(t_0)$ for some point $t_0 > \widehat{r}$ on the real line. On this disc $\mathbb{D}(t_0)$, the holomorphic function φ is represented by its Taylor series centred at t_0. Then for any $R > \widehat{R}$, using (6.27), we get:

$$\left|\varphi(t)\right| \leqslant \sum_{n=0}^{\infty} \frac{1}{n!}\left|\partial_t^n \varphi(t_0)\right| \cdot \left|t - t_0\right|^n \leqslant Ce^{Rt_0} \sum_{n=0}^{\infty} \tfrac{(n+2)!}{n!}(Mr)^n \;.$$

The infinite sum in this expression converges to a number $\tilde{C}$ independent of t. Furthermore, $t_0 < |t| + r$ because $t \in \mathbb{D}(t_0)$. So setting $A := Ce^{Rr}\tilde{C}$ yields an exponential bound $\left|\varphi(t)\right| \leqslant Ae^{R|t|}$ which is valid for all $t \in \Omega$.

To demonstrate equality (6.20), consider again the second remainder term $R_2(x) = f(x) - \big(a_0 + xa_1\big)$. For any $x \in \mathsf{S}$, let β be a Borel circle of some inverse-radius ρ such that $\mathrm{Re}(1/x) > \rho > R$. In addition, take any $0 < \varepsilon < |x|$ and consider the circle γ in $\mathbb{C}_x$ centred at 0 with radius ε, oriented clockwise. It intersects β in exactly two points; let γ_ε be the arc of γ that connects them and lies in S. Correspondingly, denote by β_ε the part of β that lies outside the disc of radius ε. Let the combined oriented contour be denoted by Γ_ε. Then we apply the Cauchy Integral Formula to $x^{-1}R_2(x)$:

$$x^{-1}R_2(x) = \frac{1}{2\pi i}\int_{\Gamma_\varepsilon} \xi^{-1}R_2(\xi)\frac{\mathrm{d}\xi}{x - \xi} = \frac{1}{2\pi i}\int_{\beta_\varepsilon + \gamma_\varepsilon} \xi^{-1}R_2(\xi)\frac{\mathrm{d}\xi}{x - \xi}$$

Thanks to the estimate (6.22), we get a bound $\left|\xi^{-1}R_2(\xi)\right| \leqslant 2CM^2|\xi|$, so the integral along the contour β_ε goes to 0 as $\varepsilon \to 0$. It follows that

$$x^{-1}R_2(x) = \frac{1}{2\pi i}\int_\beta \xi^{-1}R_2(\xi)\frac{\mathrm{d}\xi}{x-\xi} .$$

Now, we write

$$\frac{1}{x-\xi} = \frac{1}{x\xi}\cdot\frac{1}{1/\xi - 1/x} = \frac{1}{x\xi}\int_0^{+\infty} e^{u/\xi - t/x}\,\mathrm{d}t .$$

which gives:

$$x^{-1}R_2(z) = \frac{x^{-1}}{2\pi i}\int_\beta\int_0^{+\infty} R_2(\xi)e^{t/\xi - t/x}\,\mathrm{d}t\,\frac{\mathrm{d}\xi}{\xi^2} .$$

Using the estimate (6.22) again, we can apply Fubini's Theorem in order to finally obtain

$$R_2(x) = \int_0^{+\infty} e^{-t/x}\left(\frac{1}{2\pi i}\int_\beta R_2(\xi)e^{t/\xi}\frac{\mathrm{d}\xi}{\xi^2}\right)\mathrm{d}t .$$ □

The proof of Lemma 6.3 is now complete. ■