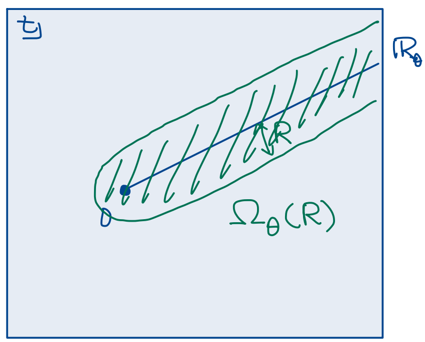
Thm (Formal Borel-Laplace Isomorphism):

$$\begin{tikzcd}
x\,\mathbb{C}[\![x]\!] \arrow[rr, bend left, "\widehat{\mathcal{B}}"] \arrow[rr, phantom, "\cong"] & & \mathbb{C}[\![t]\!] \text{ with } * \arrow[ll, bend left, "\widehat{\mathcal{L}}"] & \text{as algebras} \\
x\,\mathbb{C}^1[\![x]\!] \arrow[u, hook] \arrow[rr, bend left, "\widehat{\mathcal{B}}"] \arrow[rr, phantom, "\cong"] & & \mathbb{C}\{t\} \text{ with } * \arrow[u, hook] \arrow[ll, bend left, "\widehat{\mathcal{L}}"] & \text{as algs}
\end{tikzcd}$$

- Want to extend this to asymptotics with factorial growth. Due to Watson, Nevanlinna, and Sokal

§6. Borel Resummation



- $\forall \theta$, a <u>halfstrip</u> around $\mathbb{R}_\theta$ is

$$\Omega_\theta := \{t \mid \operatorname{dist}(t, \mathbb{R}_\theta) < R\}$$

- $\mathcal{E}^1(\Omega_\theta) := \left\{ \begin{array}{c} \text{functions of exponential} \\ \text{type at } \infty \end{array} \right\}$

  i.e. $\varphi \in \mathcal{O}(\Omega_\theta)$ st

$$|\varphi(t)| \leq C e^{M|t|} \quad \forall t \in \Omega_\theta$$

- $\mathcal{E}^1_{(\theta)} := \{\text{their germs}\} = \{ (\varphi, \Omega_\theta) \sim (\varphi', \Omega'_\theta) :\Leftrightarrow \varphi = \varphi' \text{ on } \Omega_\theta \cap \Omega'_\theta \}$

- $\mathcal{A}^1_A := \{ f \mid f^{(k)}$ bdd with fact. growth $\forall A' \Subset A \}$
  i.e. $|f^{(k)}/k!| \leq C M^k (k!)$

  $\cup$ subalgebra

  $\underline{\mathcal{A}}^1_A := \{ f \mid$ —— " —— " —— " —— <u>uniformly</u> $\forall A' \Subset A \}$.

<u>Thm</u> (<u>Borel-Laplace Isomorphism</u>)

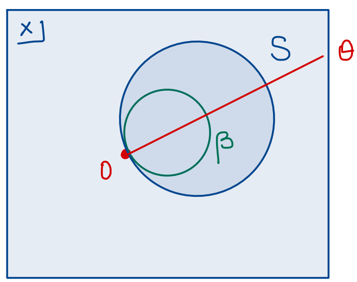
Borel and Laplace transforms restrict to mutually inverse alg. isos:

$$\begin{tikzcd}
\underline{\mathfrak{A}}^1_A \arrow[r, bend left, "\mathcal{B}_\theta"] \arrow[r, phantom, "\cong"] \arrow[d, "\text{æ}"'] & \mathcal{E}^1_{(\theta)} \arrow[l, bend left, "\mathcal{L}_\theta"] \arrow[d, "J"] \\
x\mathbb{C}^1[[x]] \arrow[r, bend left, "\hat{\mathcal{B}}"] \arrow[r, phantom, "\cong"] & \mathbb{C}\{t\} \arrow[l, bend left, "\hat{\mathcal{L}}"]
\end{tikzcd}$$

$A := (\theta - \pi/2, \theta + \pi/2)$

For $f \in \underline{\mathcal{A}}^1(S)$

$$\mathcal{B}_\theta[f] := \frac{1}{2\pi i} \underbrace{\text{P.V.}}_{\text{Cauchy principal value}} \int_\beta e^{t/x} f(x) \frac{dx}{x^2}$$



$\beta = \{ \operatorname{Re}(e^{i\theta}/x) = 1/R \}$

- Vertical arrows are <u>not</u> isomorphisms, let's fix that.

Def: $\hat{\varphi} \in \mathbb{C}\{t\}$ has <u>endless analytic continuation of exponential type in the direction θ</u> if it admits an. cont. $\varphi \in \mathcal{E}^1(\Omega_\theta)$ for some Ω_θ.

↳ they form a subalg. $\mathbb{C}^1_\theta\{t\} \subset \mathbb{C}\{t\}$

Def: $\hat{f} \in \mathbb{C}^1[\![x]\!]$ is <u>Borel summable in the direction θ</u> if $\hat{\mathcal{B}}[\hat{f}] \in \mathbb{C}^1_\theta\{t\}$

↳ they form a subalg. $\mathbb{C}^1_\theta[\![x]\!] \subset \mathbb{C}^1[\![x]\!]$.

Thm: (<u>Borel Resummation Theorem</u>)

Let $A = (\theta - \pi/2, \theta + \pi/2)$.

Then $\ae$ restricts to algebra isomorphism

$$\mathcal{A}^1_A \underset{\Sigma_\theta}{\overset{\ae}{\rightleftarrows}} \mathbb{C}^1_\theta[\![x]\!] \qquad (\cong)$$

Its inverse Σ_θ is called <u>Borel resummation</u> in the direction θ.

Cor: (<u>Strong Watson's Theorem</u>)

Consider $\ae : \mathcal{A}^1_A \longrightarrow \mathbb{C}^1[\![x]\!]$.

① If $|A| > \pi$, then $\ae$ is inj., but not surj.

② If $|A| \leq \pi$, then $\ae$ is surj., but not inj.

$\Rightarrow$ $\ae$ is <u>not</u> an isomorphism for any arc A

• if we restrict Borel Resummation iso to the ideal

Cor: (Borel-Laplace Method)

Let $A = (\theta - \pi/2, \theta + \pi/2)$.

Then we have a commutative diagram of algebra isomorphisms:

$$
\begin{tikzcd}
\mathcal{A}^1_A \arrow[rr, bend left, "\mathcal{X}"] \arrow[d, bend left, "\mathcal{B}_\theta"] \arrow[rr, phantom, "\cong"] \arrow[d, phantom, "\cong"] & & x\,\mathbb{C}^1_\theta[[x]] \arrow[ll, bend left, "\Sigma_\theta"] \arrow[d, bend left, "\hat{\mathcal{B}}"] \arrow[d, phantom, "\cong"] \\
\mathcal{E}^1_{(\theta)} \arrow[u, bend left, "\mathcal{L}_\theta"] \arrow[rr, bend left, "J"] \arrow[rr, phantom, "\cong"] & & \mathbb{C}^1_\theta\{t\} \arrow[u, bend left, "\hat{\mathcal{L}}"] \arrow[ll, bend left, "\text{analytic continuation}"]
\end{tikzcd}
$$

$$\Rightarrow \boxed{\Sigma_\theta = \mathcal{L}_\theta \circ \begin{pmatrix}\text{an}\\ \text{cont}\end{pmatrix} \circ \hat{\mathcal{B}}}$$

<u>i.e.</u>: $\forall \hat{f} \in \mathbb{C}^1_\theta[[x]]$ $\exists!\, f \in \mathcal{A}^1_A$ with $\mathcal{X}(f) = \hat{f}$ and

$$f(x) = a_0 + \int_{\mathbb{R}_\theta} e^{-t/x} \underbrace{\varphi(t)}_{\substack{\text{an. cont. along } \mathbb{R}_\theta \\ \text{of } \hat{\varphi}(t) := \hat{\mathcal{B}}[\hat{f}](t).}}\, dt$$