Fibre Bundles and Spin Structures

Part 5: Vector Bundles

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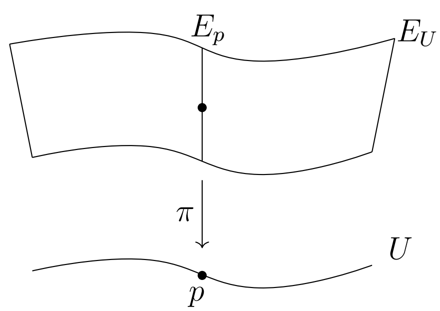
Vector Bundles are comprised of a collection of vector spaces that are smoothly parameterised by a manifold. This amounts to attaching a vector space to each point in the manifold, and then endowing this collection of attached vector spaces with a manifold structure of its own. The key idea is that locally a vector bundle resembles a Cartesian product:



whereas globally there might be nontrivial behaviour like twists. An important idea within bundle theory is that, ontologically, a bundle is nothing more than a collection of Cartesian products $U \times \mathbb{R}^k$ that are glued together.

This week we explore the formal details of the above picture. We will start with some general theory and constructions, and then we will move on to the particular case of the tangent bundle of a smooth manifold.

Contents

1 Vector Bundles

A vector bundle is a collection of vector spaces that are smoothly attached to a manifold. Vector bundles are defined so as to generalise the Cartesian product $M \times \mathbb{R}^k$. We allow vector bundles look a Cartesian product *locally*, whereas globally they may not.

Before getting to the formal definition, we should note that different authors define bundles in slightly different ways. They all end up being equivalent, but there is a cloud of possible starting properties that all imply each other. Therefore, what we might write as a definition could be a provable result for some others.

1.1 Basic Notions

The formal definition of a vector bundle may seem obscure at first, but it essentially all that is required to formally capture the intuition described previously. The definition is as follows.

Definition 1.1. *A vector bundle of rank k is a tuple (E, π, M), where E and M are smooth manifolds, and $\pi : E \to M$ is a smooth, surjective map satisfying the following properties.*

1. *For each p in M, the pre-image $\pi^{-1}(p) \subset E$ is a k-dimensional real valued vector space, and*

2. *For every p in M there is a neighbourhood U of p and a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}$ such that the following diagram commutes (where p_1 is the map that projects onto the first factor),*

$$\begin{CD} \pi^{-1}(U) @>{\Phi}>> U \times \mathbb{R}^k \\ @V{\pi}VV @VV{p_1}V \\ U @= U \end{CD}$$

and for each $q \in U$, the restriction $\Phi|_q : E_q \to \{q\} \times \mathbb{R}^k$ is an isomorphism of vector spaces.

The map Φ is known as a *local trivialisation*, and allows us to compute a local expression of objects of E in terms of vector spaces. These are analogous to the open charts of a smooth manifold, and there is also an associated notion of compatibility between local trivialisations. We will explore this more in Section 1.3.

For now, we complete this section by discussing what it means for a map between bundles to be structure-preserving. Such maps are called *bundle morphisms*, and are defined as follows.

Definition 1.2. *Let $f : M \to N$ be a smooth map, and let (E, π_E, M) and (F, π_F, N) be vector bundles. A smooth map $g : E \to F$ is said to be a bundle morphism covering f iff the diagram*

$$\begin{array}{ccc} E & \xrightarrow{\;g\;} & F \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi_F} \\ M & \xrightarrow[\;f\;]{} & N \end{array}$$

commutes, and g is a fibrewise linear map.

Observe that a bundle morphism is a smooth map (i.e. a morphism of manifolds) and a linear map (i.e. a morphism of vector spaces) once restricted to fibres. The requirement that the map covers f amounts to requiring that g maps each fibre E_p to $F_{f(p)}$. Furthermore, a bundle morphism g is called a *bundle isomorphism* iff it is bijective and its inverse map g^{-1} is also a bundle morphism. We have the following useful condition for identifying bundle isomorphisms.

Lemma 1.3. *Let E and F be vector bundles over the same base manifold M, and $g : E \to F$ be a bundle morphism covering the identity map on M. If g is bijective, then g is a bundle morphism.*

1.2 Sections and Frames

We will now define the notion of a section of a bundle. This is fairly straightforward – a section of a vector bundle E is formed by smoothly choosing a single element from each fibre E_p, as in Figure 1. Formally, sections are defined as follows.

Definition 1.4. *Let E be a vector bundle over M. A section is a smooth map $s : M \to E$ such that $\pi \circ s = id_M$. We denote the space of all sections of E by $\Gamma(E)$.*

We can similarly define a *local* section by replacing M by an open subset U of M. Under this terminology, we will refer to the sections defined in 1.4 as global sections.

1.2.1 The Structure of $\Gamma(E)$

Denote by $\Gamma(E)$ the space of all global sections of E. We will now flesh out some of the structure of $\Gamma(E)$. We first observe that $\Gamma(E)$ is closed under addition. Indeed, suppose that s_1 and s_2 are two global sections of E. Then the sum $(s_1 + s_2) : M \to E$ is given by

$$(s_1 + s_2)(p) := s_1(p) + s_2(p),$$

where on the right hand side addition is given as the vector space addition in the fibre E_p. One can confirm that the above expression defines a smooth global section, and thus $(s_1 + s_2) \in \Gamma(E)$.

We can also scale sections by smooth functions $f \in C^\infty(M)$:

$$(fs)(p) := f(p)s(p),$$

where on the right-hand side we use scalar multiplication in the fibre E_p. We may therefore conclude that the space of global sections $\Gamma(E)$ is a module over $C^\infty(M)$.

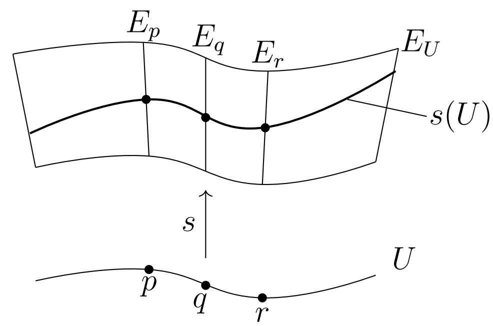


Figure 1: A smooth local section s of E_U

Remark 1.5. *At this point you should really be feeling a strong sense of deja vu. We have made all of these arguments previously when discussing tensor fields on M in Week 3. This is no coincidence – the tensor fields previously discussed are simply global sections of an appropriate tensor bundle defined on M.*

1.2.2 Frames

In the context of linear algebra, a frame of a vector space is an ordered basis. We can extend this notion to vector bundles by using sections.

Definition 1.6. *A local frame of E is a collection of k-many local sections $s_i : U \to E_U$ such that for each p in U, the collection $(s_1(p), ..., s_k(p))$ is a frame for E_p.*

Included in the above definition is the idea that a local frame consists of fibrewise linearly independent local sections. We can also extend the terminology by saying that a global frame is a local frame which is defined on $U = M$.

An important feature of frames is that they encode the same information as a local trivialisation. Indeed, suppose that we have a local frame s_i of E defined on some open set U of M. We can define a function $\Phi : E_U \to U \times \mathbb{R}^k$

that acts by sending the frame s_i to the canonical basis e_i of $\mathbb{R}^k$. Conversely, if we have a local trivialisation map $\Phi : E_U \to U \times \mathbb{R}^k$, then we can define a local frame $s_i : U \to E_U$ by

$$s_i(p) := \Phi^{-1}(p, e_i).$$

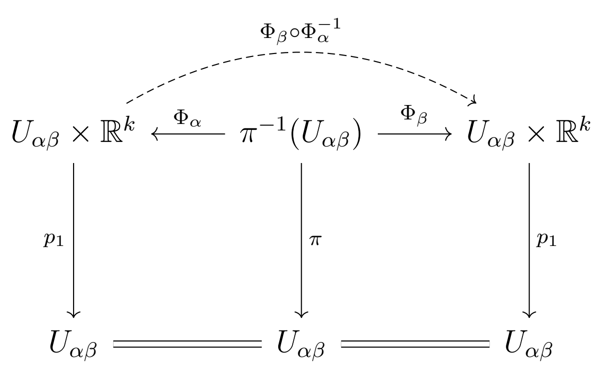
This will be a frame on each fibre E_p since the map Φ is an isomorphism of fibres. With all of this in mind, we can rephrase the local triviality condition of Definition REF by asserting that every point p in M has some open neighbourhood U on which admits a local frame of E. As a corollary of this idea, we have the following useful result.

Theorem 1.7. *A rank k vector bundle E is trivial if and only if E admits k-many global sections that are non-zero and are linearly independent on each fibre.*

The above theorem is just saying that a vector bundle is trivial if it admits a global frame.

1.3 Reconstruction of a Vector Bundle

Suppose that we have two local trivialisations (U_α, Φ_α) and (U_β, Φ_β) such that the intersection $U_\alpha \cap U_\beta$ is non-empty. We then have the following diagram,



where we have used $U_{\alpha\beta}$ as a shorthand for the intersection $U_\alpha \cap U_\beta$. Since both Φ's are bundle isomorphisms, the map $\Phi_\beta \circ \Phi_\alpha^{-1}$ is well-defined, and sends $(p, r) \mapsto (p, g_{\alpha\beta}(p)(r))$, where the map $g_{\alpha\beta}$ maps each element p of $U_{\alpha\beta}$ to some linear transformation on $\mathbb{R}^k$. In fact, it is a map $g_{\alpha\beta} : U_{\alpha\beta} \to GL_k(\mathbb{R})$, where $GL_k(\mathbb{R})$ is the *general linear group of degree* k over $\mathbb{R}$. This is the set of all invertible $k \times k$ matrices with entries in $\mathbb{R}$, and is a Lie group. The Lie group associated to the maps $g_{\alpha\beta}$ is known as the *structure group* of the bundle E. The maps $g_{\alpha\beta}$ are known as *transition functions*, and are similar to the transition maps of compatible charts of a smooth manifold. We also have the following useful result, which allows to construct vector bundles from local data.

Lemma 1.8 (Bundle Chart Lemma)**.** *Let M be a smooth manifold, and suppose that for each p in M we are given a real-valued vector space E_p of some fixed dimension k. Let $E := \bigsqcup_p E_p$, and let $\pi : E \to M$ be the map that takes each element of E_p to the point p. Suppose furthermore that we are given the following data:*

1. *an open cover $\{U_\alpha\}_{\alpha \in A}$ of M*
2. *for each $\alpha \in A$, a bijection $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ whose restriction to each E_p is a vector space isomorphism from E_p to $\{p\} \times \mathbb{R}^k$*
3. *for each $\alpha, \beta \in A$ with $U_{\alpha\beta} \neq \emptyset$, a smooth map $g_{\alpha\beta} : U_{\alpha\beta} \to GL_k(\mathbb{R})$ such that the map $\Phi_\alpha \circ \Phi_\beta^{-1}$ from $U_{\alpha\beta}$ to itself has the form $\Phi_\alpha \circ \Phi_\beta^{-1}(p, v) = (p, g_{\alpha\beta}(p)(v))$.*

Then E has a unique topology and smooth structure making it into a vector bundle over M, with π as its projection map, and (U_α, Φ_α) as local trivialisations.

For the full proof, the reader is invited to read Lee Chapter 10.

Since local trivialisations are the same as local frames, there should be an analogous derivation of transition maps using the language of frames. This is indeed true – suppose that $s_i : U_\alpha \to E_{U_\alpha}$ and $\tilde{s}_i : U_\beta \to E_{U_\beta}$ are two local frames, and suppose that $U_{\alpha\beta}$ is non-empty. Then for each point p in $U_{\alpha\beta}$, the two frames s^i and $\tilde{s}^i$ can be related to eachother by a change of basis map:

$$s^i = \lambda^i_j(p)\tilde{s}^j(p),$$

where λ^i_j is some set of coefficients that depends smoothly on p. As such, this λ^i_j can be seen as a map from $U_{\alpha\beta}$ into $GL(k)$. A quick computation will verify that the maps λ^i_j are precisely equal to the transition maps $g_{\alpha\beta}$.

2 Vector Bundle Constructions

We have seen that we can reconstruct vector bundles from local trivialisations, which we phrased in terms of transition functions and in terms of local frames. In this section, we will take this a step further by showing how to construct new bundles from old data. The idea underpinning all of the constructions is that we perform a linear-algebraic construction fibrewise, and then lift this to a globally-defined vector bundle using the (re)construction theorem.

2.1 Subbundles and Quotient Bundles

Suppose that we have a vector space V and a subspace W. We can quotient V by W to form a lower-dimensional vector space defined as follows.

$$V/W = V/\sim \text{ where } v \sim v' \text{ iff } v - v' \in W.$$

The resulting space is a vector space of dimension $dim(V) - dim(W)$.

We would like to pass this idea over to vector bundles. In order to do so, we first need to make precise the notion of a subbundle.

Suppose that we have a vector bundle E of rank k. A subbundle F of E is a collection of vector spaces F_p such that F_p is a vector subspace of E_p for all p, with the dimension of each F_p being the same. Moreover, the collection F of all F_p comprises a vector bundle in its own right, in such a way that F is a submanifold of E. Since E is a vector bundle, we expect that we can relate the transition functions of E to the transition functions of its subbundles. This is easier to state in terms of local frames.

Proposition 2.1. *Let E be a rank-k vector bundle over M and suppose that for each p in M we have an l-dimensional subspace F_p of E_p. The collection $F := \bigsqcup_p F_p$ is a rank-l subbundle of E if and only if for each p there exists a collection of l-many local sections $s_i : U \to E$ such that for each q in U, the collection $(s_1(q), ..., s_k(q))$ forms an ordered basis of F_q.*

This proposition is stating that the collection F of fibrewise subspaces is a subbundle whenever we can find a local frame for F around each point.

The quotient of a vector bundle E by one of its subbundles F is a new vector bundle ${}^{E}\!/\!{}_{F}$ whose fibres consist of the vector space quotients, that is

$$\left({}^{E}\!/\!{}_{F}\right)_p = {}^{E_p}\!/\!{}_{F_p}.$$

The projection map is obvious. The transition functions $h_{\alpha\beta}$ of the quotient bundle ${}^{E}\!/\!{}_{F}$ can be described by arranging the transition functions of E have the block form

$$g^E_{\alpha\beta} = \left(\begin{array}{c|c} g^F_{\alpha\beta} & - \\ \hline 0 & h_{\alpha\beta} \end{array}\right)$$

where $g^F_{\alpha\beta}$ is an $l \times l$ block of transition data for F, and $h_{\alpha\beta}$ is the $(k-l)\times(k-l)$ block of transition data for ${}^{E}\!/\!{}_{F}$. The idea behind this rearrangement is to use the local frame of Proposition 2.1 and extend it to a local frame for E. For the details, see Taubes Section 4.2.

2.2 Algebraic Bundle Constructions

We will now describe the bundle-theoretic analogues to various linear algebraic constructions. Suppose that E and F are vector bundles over M of rank k and l respectively. The idea behind these constuctions is to perform the construction fibrewise, and then to use the transition maps of E and F to create transition maps for bundles of higher ranks. The new bundles will then exist by an application of 1.8.

1. **The Dual Bundle**. The dual bundle E^* of E consists of the following data.

   - Fibres: the dual spaces of the fibres of E, that is, $(E^*)_p = (E_p)^*$.
   - Transition maps: $g^*_{\alpha\beta} : U_{\alpha\beta} \to GL(k)$ given by $g^*_{\alpha\beta} = ((g_{\alpha\beta})^T)^{-1}$, where $g_{\alpha\beta}$ are the transition functions of E.

2. **The Direct Sum**. The direct sum $E \oplus F$ of the bundles E and F consists of the following data.

   - Fibres: the direct sum spaces of the fibres of E and F, that is, $(E \oplus F)_p = E_p \oplus F_p$.
   - Transition maps: $g_{\alpha\beta} : U_{\alpha\beta} \to GL(k+l)$ given by $g_{\alpha\beta} = g^E_{\alpha\beta} \oplus g^F_{\alpha\beta}$, that is
   $$g_{\alpha\beta} = \left(\begin{array}{c|c} g^E_{\alpha\beta} & 0 \\ \hline 0 & g^F_{\alpha\beta} \end{array}\right).$$

3. **The Tensor Product**. The tensor product $E \otimes F$ of the bundles E and F consists of the following data.

   - Fibres: the direct sum spaces of the fibres of E and F, that is, $(E \oplus F)_p = E_p \oplus F_p$.
   - Transition maps: $g_{\alpha\beta} : U_{\alpha\beta} \to GL(kl)$ given by $g_{\alpha\beta} = g^E_{\alpha\beta} \otimes g^F_{\alpha\beta}$, which is the tensor product of matrices once applied to each p in $U_{\alpha\beta}$.

4. **The Hom-Bundle**. The space of linear maps between vector spaces V and W is denoted by $Hom(V,W)$.[1] As a basic fact of Linear algebra, there is an isomorphism $Hom(V,W) \cong V^* \otimes W$. We can perform this construction on vector bundles to create $Hom(E,F)$, whose fibres consist of the linear maps between fibres of E and F, that is, $(Hom(E,F))_p = Hom(E_p,F_p)$. The transition functions can be described by viewing the Hom-bundle as $E^* \otimes F$.

5. **The Endomorphism Bundle**. An endomorphism of a vector space V is a linear map to itself. We write the collection of all endomorphisms of V as $End(V)$. This space is equal to $Hom(V,V)$, that is, $V^* \otimes V$, the rank $(1,1)$-tensors. We will use the same notation $End(E)$ to denote the bundle $E^* \otimes E$.

2.3 Restrictions and Pullbacks

Given a smooth map $f : M \to N$ and a vector bundle (F, π_F, N), we can define a new bundle over M by restricting the structure of F to the fibres that cover $f(M)$. This is called a *pullback bundle*, and is commonly denoted by f^*F. The fibres are given by $(f^*F)_p = F_{f(p)}$. The formal definition is as follows.

Definition 2.2. *Let $f : M \to N$ be a smooth map and (F, π_F, N) a vector bundle. The pullback bundle has as elements:*

$$f^*F := \{(m,u) \in M \times F \mid f(m) = \pi_F(u)\},$$

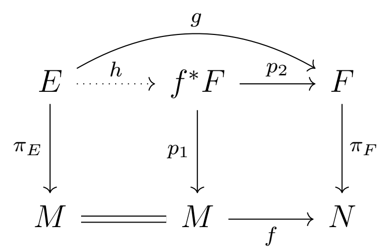
and the projection map of f^*F is given by the projection onto the first factor.

We also make the following useful observation.

Proposition 2.3. *Let $f : M \to N$ be a smooth map, and (F, π_F, N) a vector bundle. The map $p_2 : f^*F \to F$ is a bundle morphism, and p_2 is an isomorphism whenever f is a diffeomorphism.*

[1] The "hom" is short for "homomorphism", the general term.

It can also be shown that every bundle morphism $g : E \to F$ that covers the map f has to factor through the pullback bundle f^*F, i.e. for every such g there is a unique morphism $h : E \to f^*F$ such that the following diagram commutes.



Given a vector bundle (E, π, M) and an embedded submanifold A of M, we can restrict the bundle E to A by pulling E back along the inclusion map $\iota : A \to M$ (which is smooth by definition). We will denote this bundle by $E|_A$, instead of ι^*E. The bundle $(E|_A, \pi|_A, A)$ is then called a *restricted bundle*.

3 The Tangent Bundle

The tangent bundle of a smooth manifold is the prototypical example of a vector bundle. The reason for this is that the structure naturally follows from the properties of a smooth manifold so it is quite easy to introduce. In this section we will provide the details of the construction of the tangent bundle.

3.1 Constructing the Tangent Bundle

It is possible to define a vector-bundle structure on the set

$$TM := \bigsqcup_{p \in M} T_pM.$$

There is a natural projection map $\pi : TM \to M$ which sends every tangent vector in T_pM to p. We can endow this collection of tangent spaces with a manifold structure by considering a local chart of M. Suppose we have a chart (U, φ) for M, and consider the coordinate functions $x^i : U \to \mathbb{R}$. We would like to define a map $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$. Let v be some element of $\pi^{-1}(U)$. This means that v sits inside some tangent space T_pM, for some point p in U. Since U is an open chart of M we can express v in the coordinate basis induced from U, that is:

$$v = v_i \frac{\partial}{\partial x^i},$$

where we suppress the dependency on p for simplicity. The map $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ can then defined by

$$\Phi(v) = \Phi\left(v_i \frac{\partial}{\partial x^i}\right) = (p, v_1, ..., v_n),$$

that is, we send every vector v to its components in the coordinate basis induced from U.

We can go one step further and use the map Φ to describe a chart for TM using $\pi^{-1}(U)$, by mapping $(p, v_i \frac{\partial}{\partial x^i}) \mapsto (x^1(p), ..., x^n(p), v_1, ..., v_n)$. This can be summarised by the diagram:

$$\begin{array}{ccccc} TM|_U & \xrightarrow{\Phi} & U \times \mathbb{R}^m & \xrightarrow{(\varphi, id)} & \varphi(U) \times \mathbb{R}^m \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_1} \\ U & = & U & \xrightarrow[\varphi]{} & \varphi(U) \end{array}$$

where here we use (φ, id) as shorthand for the map which acts by φ on the first coordinate of $U \times \mathbb{R}^k$ and does nothing to the others.

Remark 3.1. *Recall that vector fields are smooth assignments of tangent vectors to each point in M. In the language of vector bundles, a vector field v in $\mathfrak{X}(M)$ is actually a global section of the tangent bundle, that is $v : M \to TM$ is a smooth right inverse of the projection map π.*

3.2 The Differential Revisited

It is also possible to define the differential of a smooth map $f : M \to N$ using the pointwise differentials df_p. The differential of f is denoted by df, and is defined as

$$df(p, v) = (f(p), df_p(v)).$$

We have the following useful facts about the differential map.

Proposition 3.2. *If $f : M \to N$ is a smooth map, then the differential $df : TM \to TN$ is smooth.*

Sketch. We compute the local representation of df in terms of charts of TM and TN. For local charts (U, φ) of M and (V, ψ) of N, we obtain the following diagram.

$$\begin{array}{ccccccc}
\varphi(U) \times \mathbb{R}^m & \xleftarrow{\Phi} & TM|_U & \xrightarrow{df} & TN|_V & \xrightarrow{\Psi} & \psi(V) \times \mathbb{R}^n \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p_1} \\
\varphi(U) & \xleftarrow[\varphi]{} & U & \xrightarrow[f]{} & V & \xrightarrow[\psi]{} & \psi(V)
\end{array}$$

The key observation is that by using Euclidean space, the local representation of the differential df will be the differential of the local representation of f. By some basic computations one will see that a point (p, v) in $\varphi(U) \times \mathbb{R}^m$ will be mapped to $(\psi \circ f \circ \varphi^{-1}(p), d(\psi \circ f \circ \varphi^{-1})_p(v))$ under the local representation of df. The first component is smooth since it is the coordinate representation of a smooth map, and the second component is smooth since it is the derivative of a smooth map. Since each component of this function is smooth, the entire coordinate representation of df is smooth. For the full details, see Dundas Lemma 5.5.4. □

Since the pointwise differential df_p is a linear map, we may conclude from the above that the differential df is a bundle morphism from TM into TN covering f, as organised in the following diagram.

$$\begin{array}{ccc} TM & \xrightarrow{df} & TN \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow[f]{} & N \end{array}$$

We also have the following facts, which follow routinely from the properties of the pointwise differential and bundle morphisms.

Proposition 3.3. *Let $f : M \to N$ and $g : N \to L$ be smooth maps. Then*

1. $d(g \circ f) = dg \circ df$
2. *If f is a diffeomorphism then df is a bundle isomorphism, with $(df)^{-1} = d(f^{-1})$.*
3. *If $f : M \to N$ is a diffeomorphism then $df : TM \to TN$ is a bundle isomorphism.*

3.3 Further Bundles

We can use the constructions detailed in Section 2 to construct other bundles from TM. According to our discussion in Section 2, we should be able to describe a dual bundle to TM by suitably modifying the transition functions of TM. Recall the transformation rule for cotangent vectors:

$$\tilde{\omega}_i = \omega_j \frac{\partial y^j}{\partial x^i}.$$

This map is indeed the inverse transpose of the Jacobian that is used to transform tangent vectors. A similar line of reasoning confirms the transition charts of tensor bundles are indeed the transformation laws for tensors on M.

4 Some Parallelizable Manifolds

A manifold is called parallelizable if its tangent bundle is trivial. From Theorem 1.7, this means that a manifold M is parallelizable if its tangent bundle admits a global frame, that is, if we can find n-many global sections that are linearly independent and everywhere non-vanishing. We will now review several interesting examples of manifolds and their tangent bundles.

4.1 The Tangent Bundle of S^1

We opt for an extrinsic construction of S^1, by interpreting S^1 as the set of unit norm vectors in $\mathbb{R}^2$. Based on this construction, we expect that the tangent space at a point T_pS^1 will be a ray in $\mathbb{R}^2$ that touches S^1 at precisely one point. Precisely, the tangent space T_pS^1 should be the one-dimensional subspace of $\mathbb{R}^2$ that is normal to the vector formed from p.

We can create a simple formula for the tangent space T_pS^1. Let $p = (x, y)$. Then the tangent space at p will be

$$T_pS^1 = \{(v_1, v_2) \in \mathbb{R}^2 \mid xv_1 + yv_2 = 0\}.$$

Consider the tangent bundle TS^1. There are really only two choices for the structure of this bundle – either TS^1 is equal to the infinite cylinder, or it has a twist and is thus equal to the Möbius bundle. We can show that TS^1 is trivial by constructing a non-zero section. Define $s : S^1 \to TS^1$ by

$$s(x, y) = (-y, x).$$

Clearly this lies in the correct tangent space, and moreover it is non-vanishing, since (x, y) lies on the unit norm circle. Moreover, this map is smooth. It follows that the tangent bundle TS^1 is trivial, that is $TS^1 \cong S^1 \times \mathbb{R}$.

Alternatively, we can use the complex plane $\mathbb{C}$ instead of $\mathbb{R}^2$. The non-vanishing section will be given by $s(z) = iz$, which is geometrically identical to the section described above.

4.2 The Tangent Bundle on S^2 and S^3

If we try to repeat the same trick on S^2 we are doomed to fail. This follows from the interesting classical result known as the Hairy ball theorem, originally due to Poincarè.

Theorem 4.1. *There is no continuous non-vanishing vector field on* S^2.

If we were to try and argue that the tangent bundle of S^2 is trivial, then we would need two non-vanishing global sections. But, the theorem above tells us that we cannot find one such section, let alone two! We may thus conclude that the 2-sphere is not parallelizable.

However, we *can* describe the tangent bundle of the 3-sphere explicitly. Now we want to interpret the S^3 as the sphere of unit norm sitting inside $\mathbb{R}^4$. The tangent space at a point will again be equal to the orthogonal hyperplane, that is, $T_pS^3 = p^\perp$. Fix p in S^3, and suppose that the vector corresponding to p has coordinates (x_1, x_2, x_3, x_4). Using standard techniques from linear algebra, we can describe the orthogonal complement of p as

$$T_pS^3 = \text{span}\left(\begin{pmatrix} -x_2 \\ x_1 \\ -x_4 \\ x_3 \end{pmatrix}, \begin{pmatrix} -x_4 \\ -x_3 \\ x_2 \\ x_1 \end{pmatrix}, \begin{pmatrix} -x_3 \\ x_4 \\ x_1 \\ -x_2 \end{pmatrix}\right).$$

These vectors form an orthogonal basis of T_pS^3. Moreover, they are all orthogonal to p itself.

To see that TS^3 is trivial, we can proceed as in the case of the circle. We define three sections $s : S^3 \to TS^3$ by

$$\begin{aligned} s_1(x_1, x_2, x_3, x_4) &= (-x_2, x_1, -x_4, x_3)^T \\ s_2(x_1, x_2, x_3, x_4) &= (-x_4, -x_3, x_2, x_1)^T \\ s_3(x_1, x_2, x_3, x_4) &= (-x_3, x_4, x_1, -x_2)^T, \end{aligned}$$

that is, s_i maps to each of the three orthogonal basis vectors in T_pS^3. As in the case of S^1, each of these sections is smooth and globally non-vanishing. Moreover, they are linearly independent by construction. It follows that the s_i define a global frame of TS^3, and thus TS^3 is trivial.

If we would like to be clever, we can actually interpret S^3 as the unit norm points sitting inside the Quarternions $\mathbb{H}$. For $p = h$, The tangent space T_pS^3 can then be described as

$$T_pS^3 = \text{span}(ih, jh, kh),$$

and the global frame of TS^3 can then be described by $s_1(h) = ih$, $s_2(h) = jh$ and $s_3(h) = kh$.

4.3 The Tangent Bundle of a Lie Group

We finish this section by tying in with last week's Lie theory. Recall that a Lie group G has a natural connection to a Lie algebra $\mathfrak{g}$, which can be seen as either the tangent space at the identity, or as the collection of left-invariant vector fields. To establish this equality between definitions of $\mathfrak{g}$ we used the left-translations L_g to push vectors in T_eG around the manifold to induce a smooth vector field. We can again use the left-translations to establish the following result.

Theorem 4.2. *Every Lie group is parallelizable.*

Proof. Let $(v_1, ..., v_n)$ be a frame of $\mathfrak{g}$. Recall that each vector v in $\mathfrak{g}$ induces a left-invariant vector field v^L which is defined as $v^L(g) = (L_g)_*(v)$ for all g. If we do this for the frame of $\mathfrak{g}$, we obtain n-many vector fields $(v_1^L, ..., v_n^L)$. This collection will become a global frame of TG – linear independence and global non-vanishing follows from the fact that the differential of a diffeomorphism acts as an isomorphism of vector spaces. □

4.4 Higher-Dimensional Spheres

To be completed after next week's homework.