Fibre Bundles and Spin Structures

Part 4: Vector Fields and Lie Theory

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A Lie group is a collection of symmetries that can be smoothly deformed into each other. A Lie algebra is a vector space with a particular type of anticommuting product. As the name would suggest, there is a connection between the two: the infinitesimal behaviour of a Lie group can be described using a Lie algebra.

This week we will cover some of the general properties of Lie groups and Lie algebras. In order to do so, we will need to flesh out some of the basic properties of vector fields, since we are yet to do this. As such, we will spend some time talking about vector fields, integral curves, flows and the Lie derivative. After this, we will characterise the intuitive definition of the Lie algebra as an infinitesimal approximation to a Lie group.

Whole courses are taught on Lie Theory, so sadly we only have enough time to introduce the basic material. Special attention is paid to the fragment of Lie Theory that appears in the theory of principal G-bundles and their connections. For the rest of it, you are on your own.

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1 Group Actions

A group is a structured collection of symmetries of some object. Symmetries can be composed and inverted, and the axiomatic definition of a group captures this intuition nicely.

Definition 1.1. *A group is a set G together with a binary operation $\cdot$ satisfying the following properties.*

G1) Associativity: $g \cdot (h \cdot k) = (g \cdot h) \cdot k$

G2) Identity: there exists an element e in G such that $g \cdot e = e \cdot g = g$ for all g in G

G3) Inverses: for each g in G there exists an element h in G such that $g \cdot h = h \cdot g = e$.

This definition should be familiar to all of you. What might not be familiar is that we can equivalently formulate the operation $\cdot$ and the inverse in terms of functions $m : G \times G \to G$ where $(g, h) \mapsto g \cdot h$ and $^{-1} : G \to G$ where $g \mapsto g^{-1}$.

1.1 Group Actions on a Set

Although there is usually a canonical object that is associated to a group, it is often useful to allow symmetries to act on other objects. This is known as a *group action*, and it is defined as follows.

Definition 1.2. *Let G be a group and X a set. A left action of G on X is a binary operation $\bullet : G \times X \to X$ satisfying the following conditions.*

1. *$e \bullet x = x$ for all x in X.*
2. *$g \bullet (h \bullet x) = (gh) \bullet x$ for all g, h in G and x in X.*

Whenever a set X is equipped with a left action of G, we refer to X as a left G-set.

A group action can be equivalently characterised by a group homomorphism from G to $Sym(X)$, the group of bijections of the set X. In this sense, a group action sends group transformations into permutations of X. We can therefore consider what happens to a single element x of X as it is acted on by the entirity of G. The resulting set is called the *orbit space* of x, and is defined by

$$Orb_{\bullet}(x) := \{y \in X \mid y = g \bullet x \text{ for some } g \in G\}.$$

There are two more properties of group actions that we will need when discussing principal bundles. These are *free* actions and *transitive* actions.

Definition 1.3. *Let $\bullet : G \times X \to X$ be a left action. Then*

1. *$\bullet$ is called free if for each x in X and every pair of group elements g, h, $g \bullet x = h \bullet x$ implies that $g = h$,*
2. *$\bullet : G \times X \to X$ is called transitive if $Orb_{\bullet}(x) = X$ for all x in X.*

As you might notice, a free action mimics the properties of an injective function, and transitivity asserts that the whole of X can be covered by the action of G on a single element. Observe that if the action is both free and transitive, then there is only one orbit space, that is,

$$Orb_{\bullet}(x) = Orb_{\bullet}(y)$$

for all x and y in X.

1.2 Quotients via Group Actions

Previously we have seen that a topological space can be quotiented using an equivalence relation, which was used to identify points in the space together. A useful part property of group actions is that they can be used to induce a quotient of the space that they are acting on. For now we will consider this concept at the level of quotienting topological spaces.

Suppose that we have a set X, and a group G acting on X by the left. We now define an equivalence relation $\sim$ on X that relates points as follows:

$$x \sim y \text{ iff there exists some } g \in G \text{ such that } x = g \bullet y.$$

. It is not hard to see that this relation is an equivalence relation – reflexivity and transitivity follow from the first and second properties of Def. 1.2 respectively, and symmetry follows from the existence of inverse elements. It follows that the relation $\sim$ defined above is a well-defined equivalence relation, and thus we can meaningfully form the quotient space

$$X/\sim = \{[x] \mid x \in X\}$$

. The relation $\sim$ identifies a point in X with everything in its orbit. Thus the overall space $X/\sim$ can be seen as the space of orbits of elements of X.

1.3 Quotients by Topological Groups

A topological group is a group that is also a topological space, in a "compatible" sense. This means we demand that the multiplication map m and the inverse map $^{-1}$ are continuous. If X is also endowed with a topology, we can refine our notion of group to that of a *continuous* group action. The definition is the same as that of Def. 1.2, except that now we require the map $\bullet : G \times X \to X$ to be continuous.

As an example of this, we can consider the construction of S^1 as a quotient of $\mathbb{R}$ by $\mathbb{Z}$. We define a group action $\bullet$ of $\mathbb{Z}$ on $\mathbb{R}$ as follows:

$$z \bullet r = z + r.$$

You may confirm that this is a continuous group action. The orbit spaces of this action are:

$$Orb_\bullet(r) = r + \mathbb{Z} := \{s \in \mathbb{R} \mid s = r + z \text{ for some } z \in \mathbb{Z}\}.$$

The quotient space, which we will denote by $^{\mathbb{R}}\!/\!_{\mathbb{Z}}$, therefore identifies every real number periodically. The resulting space is topologically equivalent to S^1.

2 Lie Groups

Roughly put, a Lie group is a smooth manifold that is also a group. These naturally arise in the symmetries of the circle, as well as in the autmorphisms of a vector space. In this section we will briefly review the basics of Lie groups, as well as some examples, and the notion of a Lie group action.

2.1 Basic Properties

We can use the multiplication and inverse maps of Section 1 to define a Lie group.

Definition 2.1. *A Lie group is a smooth G equipped with a group structure such that the functions $\cdot^{-1} : G \to G$ and $m : G \times G \to G$ are smooth functions.*

As one might expect, a Lie group homomorphism is a smooth map that is also a smooth manifold. Similarly, a Lie subgroup is a subgroup that is also a submanifold.

We will now introduce two important types of map, which can be obtained by fixing either argument in m. Consider:

$$L_g : G \to G,\ L_g(h) := m(g,h) = gh$$

and

$$R_g : G \to G,\ R_g(h) := m(h,g) = hg.$$

These maps are usually called the left and right translations, respectively. Since the multiplication map m is smooth, both L_g and R_g are smooth for all g in G.[1] Moreover, these actions are diffeomorphisms since they have smooth inverses given by $L_{g^{-1}}$ and $R_{g^{-1}}$, respectively. As the name would suggest, the left and right translations allow us to smoothly move between different points in the group G.

2.2 The General Linear Group

If you are a student of physics, then there is no doubt that you have already seen the matrix groups several times. However, for the sake of completeness we will introduce $GL(v)$ for the n^{th} time.

The matrix groups are Lie groups whose points are $n \times n$ matrices. Generally an emphasis is given on the group structure of these spaces, without explaining how they arise as manifolds. So, before discussing any group structures we will first consider how a collection of matrices can be made into a smooth manifold.

Let us denote the set of $n \times n$ square matrices by $\mathcal{M}^{n\times n}$. We would like to discuss the manifold structure of $\mathcal{M}^{n\times n}$. However, it is even easier to discuss the vector space structure: square matrices can be added and scaled component-wise to yield another square matrix. There is also a zero element, that is, the matrix consisting of all zeros. As such, the collection $\mathcal{M}^{n\times n}$ is actually a real-valued vector space. Since all vector spaces are globally Euclidean, they are manifolds. To determine the dimension of $\mathcal{M}^{n\times n}$, we can consider the map which concatenates the columns of a matrix into a real vector of length n^2. This rearrangement of data is clearly invertible, so $\mathcal{M}^{n\times n}$ can be seen as equal to $\mathbb{R}^{n^2}$. With this out of the way, we can now consider the first of our matrix groups.

[1] We can write $L_g = m \circ \iota_1$, where the map $\iota_1 : G \to G \times G$ maps G into the left-hand copy of G sitting inside $G \times G$.

Example 2.2 (The General Linear Group)**.** *Given a vector space V of dimension n, we can consider the collection of automorphisms of V:*

$$Gl(V) := \{A \in M^{n \times n} \mid det(A) \neq \emptyset\}.$$

The group structure is given by matrix multiplication, and the manifold structure of $GL(V)$ is induced from the manifold structure of $M^{n \times n}$.

2.3 Smooth Group Actions

In Section 1 we saw the idea of topological groups acting on another topological space continuously. We can take this one step further by considering smooth actions, which are defined as follows.

Definition 2.3. *A smooth left action of the Lie group G on a manifold M is a group action $\bullet : G \times M \to M$ that is smooth as a map between manifolds.*

A group action can also be seen as a homomorphism from G into the group $Diff(M)$ of diffeomorphisms of M, since we can map each element g of G to the map $A_g : M \to M$ defined by $A_g(m) = g \bullet m$.

Since G is itself a smooth manifold, the multiplication map $m : G \times G \to G$ fits the definition above, so can be interpreted as an action of G on itself.

2.3.1 Equivariance

Suppose we have a Lie group G that is simultaneously acting on two smooth manifolds M and N by the left. A smooth map $f : M \to N$ is said to be *equivariant* with respect to the actions of G if

$$f(g \bullet_M p) = g \bullet_N f(p)$$

for all g in G and p in M. Given an element g in G, let A_g be the action map of $\bullet_M$ and B_g be the action map of $\bullet_N$. Then a smooth map's equivariance can be characterised by saying that the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle A_g}\downarrow & & \downarrow{\scriptstyle B_g} \\ M & \xrightarrow[f]{} & N \end{array}$$

commutes.

3 Vector Fields on a Manifold

Recall that for any manifold M, a tangent vector v_p in T_pM can be defined as a derivation of the ring $C^\infty(M)$ at p, meaning that v_p could be seen as a map

$$v_p : C^\infty(M) \to \mathbb{R}.$$

A vector field v will smoothly assign a derivation v_p to each p in M. Therefore, for any smooth function f the vector field v associates a real number that smoothly depends on the points in M. Put differently, we can view a vector v as a function

$$v : C^\infty(M) \to C^\infty(M),$$

where $v(f) : M \to \mathbb{R}$ is given by $v(f)(p) := v_p(f)$. The map v can be seen as a derivation of $C^\infty(M)$, so also satisfies a global analogue of the pointwise Leibniz law. We will denote the space of all vector fields on M as $\mathfrak{X}(M)$.

3.1 Pushforwards by Diffeomorphisms

Last week we remarked that although individual tangent vectors can be pushed forward by a smooth map f between manifolds, it is not necessarily the case that a vector field can be pushed forward by a smooth map. However, there are some circumstances under which we can push forward vector fields. Perhaps the most obvious one is when the map $f : M \to N$ is a diffeomorphism.

The pushforward $f_* v$ should associate a unique tangent vector to each point q in N. If f is a diffeomorphism, then in particular it is a bijection. Therefore there is some unique element p in M such that $f(p) = q$. We can therefore use the pointwise differential $df_p : T_pM \to T_qN$ to pushforward the tangent vector v_p in T_pM to something in T_qN. In symbols, the vector field $f_* v$ associates to each q in N a tangent vector

$$(f_* v)(q) = (df)_{f^{-1}(q)}(v_{f^{-1}(q)}).$$

It can be shown that the smooth properties of the inverse $f^{-1} : N \to M$ guarantee that the vector field $f_* v$ is smooth.

In the case that our smooth map f is a diffeomorphism from M to M, the pushforward can be seen as an operation $f_* : \mathfrak{X}(M) \to \mathfrak{X}(M)$, which acts on points in M by

$$(f_* v)(p) = (df)_{f^{-1}(p)}(v_{f^{-1}(p)}).$$

This acts on functions g in $C^\infty(M)$ by:

$$(f_* v)(p)(g) = (df)_{f^{-1}(p)}(v_{f^{-1}(p)})(g) = v_{f^{-1}(p)}(g \circ f),$$

which can be rephrased as

$$f_* v(g) = v(g \circ f) \circ f^{-1}. \tag{1}$$

3.2 The Commutator of Vector Fields

Suppose now that we have two vector fields v and w. Since both are maps from $C^\infty(M)$ to itself, it makes sense to consider the composition

$$v \circ w : C^\infty(M) \to C^\infty(M).$$

This is a composition of linear maps, so it is also linear. Therefore, one might expect that the composition of two vector fields is a vector field. However, this is not the case, because the Leibniz law fails:

$$\begin{aligned} v \circ w(fg) &= v(w(f)g + fw(g)) \\ &= v \circ w(f)g + fv \circ w(g) + w(f)v(g) + v(f)w(g). \end{aligned}$$

The two extra terms (which locally will consist of mixed partial derivatives), spoil the Leibniz law. However, using inverses and compositions we can create a clever combination of vector fields that is again a derivation of $C^\infty(M)$. This is known as the *commutator*, and is defined as

$$[v, w] := v \circ w - w \circ v.$$

This is again a vector field, because the cross-terms of both $v \circ w$ and $-w \circ v$ will cancel out to ensure the Leibniz law holds (since mixed partial derivatives commute).

We can also express the commutator in local coordinates as follows. Suppose that (U, φ) is a local chart with coordinate functions x^i. Recall that the local expression of vector fields v and w will be:

$$v = v^i \frac{\partial}{\partial x^i} \text{ and } w = w^j \frac{\partial}{\partial x^j},$$

where the components v^i and w^j are members of $C^\infty(U)$. We can use a smooth function f to see how the commutator behaves locally:

$$\begin{aligned} (vw - wv)f &= v^i \frac{\partial}{\partial x^i}\left(w^j \frac{\partial f}{\partial x^j}\right) - w^i \frac{\partial}{\partial x^i}\left(v^j \frac{\partial f}{\partial x^j}\right) \\ &= \left(v^i \frac{\partial w^j}{\partial x^i}\frac{\partial f}{\partial x^j} + w^j \frac{\partial^2 f}{\partial x^i \partial x^j}\right) - (v \leftrightarrow w) \\ &= v^i \frac{\partial w^j}{\partial x^i}\frac{\partial f}{\partial x^j} - w^i \frac{\partial v^j}{\partial x^i}\frac{\partial f}{\partial x^j} \\ &= \left(v^i \frac{\partial w^j}{\partial x^i} - w^i \frac{\partial v^j}{\partial x^i}\right)\frac{\partial f}{\partial x^j}. \end{aligned}$$

We can now remove the particular function f to conclude that locally

$$[v, w] = \left(v^i \frac{\partial w^j}{\partial x^i} - w^i \frac{\partial v^j}{\partial x^i}\right)\frac{\partial}{\partial x^j}. \tag{2}$$

We will finish our discussion of the commutator by observing how it is preserved during pushforwards by diffeomorphisms.

Lemma 3.1. *Let $f : M \to M$ be a diffeomorphism and let v, w in $\mathfrak{X}(M)$. Then*

$$f_*[v, w] = [f_*v, f_*w].$$

Proof. We will fix a point p in M and see how the commutator $f_*[v, w]_p$ acts on a smooth function $g \in C^\infty(M)$. We have:

$$\begin{aligned}
f_*[v, w]_p(g) &= [v, w]_p(g \circ f) \\
&= v_p(w(g \circ f)) - w_p(v(g \circ f)) \\
&= v_p(f_*w(g) \circ f) - w_p(f_*v(g) \circ f) \\
&= (f_*v)_p(f_*w(g)) - (f_*w)_p(f_*v(g)) \\
&= \left((f_*v)_p f_*w - (f_*w)_p f_*v\right)(g) \\
&= [f_*v, f_*w]_p(g).
\end{aligned}$$

Note that in the third line we have used an alternate form of equation (1). □

3.3 Integral Curves and Flows

Suppose that we have a vector field v on M. An integral curve of v is a smooth curve γ on M whose velocity vectors equal the values of v. In symbols, a curve $\gamma : D \to M$ is an integral curve of v if

$$\gamma'(t) = v_{\gamma(t)}.$$

Note that the domain D of γ need not be all of $\mathbb{R}$. Also, all integral curves are smooth by definition.

Suppose we have a local chart (U, φ) with coordinates x^i. Then locally we can represent the above equality in the coordinate basis of $T_{\gamma(t)}M$ as

$$\frac{d\gamma^i(t)}{dt} \frac{\partial}{\partial x^i}\bigg|_{\gamma(t)} = v^i(\gamma(t)) \frac{\partial}{\partial x^i}\bigg|_{\gamma(t)}. \tag{3}$$

This equality amounts to a system of autonomous ordinary differential equations:

$$\dot{\gamma}^1(t) = v^1\Big(\gamma^1(t), ..., \gamma^n(t)\Big)$$
$$\vdots$$
$$\dot{\gamma}^n(t) = v^n\Big(\gamma^1(t), ..., \gamma^n(t)\Big),$$

where the upper dots represent the usual time derivatives. Using the above representation, we can see that finding an integral curve of v that passes through a given point p amounts to solving the above system, with initial conditions specified by $\gamma(0) = p$. Fortunately, there are theorems that guarantee the existence, uniqueness and smoothness of solutions to autonomous ODEs.[2] When applied to the above coordinate representation, we obtain the following result.

Theorem 3.2. *Let V be a smooth vector field on M. For each p in M there exists a smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ that is an integral curve with $\gamma(0) = p$.*

So, by the magic of ODEs we have the guaranteed local existence of an integral curve passing through every point in the manifold. Note that we can play around with reparametrisations of an integral curve γ to obtain other integral curves. These can either be integral curves of another vector field, or integral curves starting at different points in M. Suppose that we have an integral curve γ with domain D, and consider the following reparametrizations of γ.

- *Rescalings*: let $\tilde{\gamma}$ be a curve defined by $\gamma(t) = \gamma(at)$, with domain $\tilde{D} = \{t \mid at \in D\}$. Then γ is an integral curve of the vector field av, starting at p.

- *Translations*: let $\hat{\gamma}$ be a curve defined by $\hat{\gamma}(t) = \gamma(t + b)$, with domain $\hat{D} = \{t \mid t + b \in D\}$. Then $\hat{\gamma}$ is an integral curve of the vector field v, starting at $\gamma(b)$.

[2]See Lee Appendix D for a summary.

These are not hard to prove, we simply compute the derivatives of the new curves in local coordinates, and verify that they satsify equation (3). The results follow from an application of the chain rule and the fact that γ is an integral curve. We also have the following result regarding the pushforward of vector fields.

Lemma 3.3. *Let γ be an integral curve of v starting at p. If $f : M \to M$ is a diffeomorphism, then the composition $f \circ \gamma$ is an integral curve of $f_* v$ starting at $f(p)$.*

A useful feature of integral curves is that they induce a family of diffeomorphisms which act by "sliding" the manifold along the integral curves of some fixed vector field. Suppose that v is a vector field such that for each p in M there is a unique integral curve $\gamma^p : \mathbb{R} \to M$ starting at p.[3] For each t in $\mathbb{R}$, we can define a function $\theta_t : M \to M$ as

$$\theta_t(p) = \gamma^p(t),$$

that is, the map θ_t sends p along its integral curve for time t.

With a bit of thought, one can see that the equality $\theta_t \circ \theta_s(p) = \theta_{t+s}(p)$ holds.[4] Therefore, the collection of θ maps can be seen as an action $\theta : \mathbb{R} \times M \to M$ by the additive group $(\mathbb{R}, +)$. Indeed, we have that

$$\begin{aligned} \theta(0, p) &= \theta_0(p) = p \\ \theta(t+s, p) &= \theta_{t+s}(p) = \theta_t \circ \theta_s(p). \end{aligned}$$

Since the integral curves are smooth, this is a smooth group action. This motivates the following definition.

[3]It is not always the case that an integral curve has domain equal to the whole of $\mathbb{R}$. Actually, this is known as a *maximal* integral curve. We suppose it for illustration purposes.

[4]This observation relies on our assumption that γ^p is unique and maximal. Indeed, the integral curve $\gamma^{\theta_s(p)}$ can be described as a translation of γ^p, where we translate the parameter t by $t + s$.

Definition 3.4. *A global flow on* M *is a smooth group action of* $(\mathbb{R}, +)$ *on* M*.*

Alternatively, global flows are sometimes called *one-parameter group actions* on M. According to the discussion so far, we have seen that flows can be defined according to the (maximal) integral curves of some vector field. It is not the case that every vector field generates a global flow. This is because the integral curves of the vector field may not have domains across the whole of $\mathbb{R}$. However, it can be shown that *every* global flow is generated by some vector field.

According to these observations, we would like to consider the particular collection of vector fields that generate flows. This is captured with the following terminology.

Definition 3.5. *A vector field is called complete if it generates a global flow.*

Some authors define a complete vector field to be one that has all integral curves with domain equal to $\mathbb{R}$. Based on what we have discussed here, it should be clear that these two definitions are equivalent.

3.4 The Lie Derivative

The idea behind the Lie derivative is that it is the correct generalisation of the directional derivative to manifolds. Recall that in Euclidean space the directional derivative of a function f in direction given by a vector v is given by

$$D_v f = \lim_{t \to 0} \frac{f(p + tv) - f(p)}{t}.$$

We are comparing a tiny change in the function f, where the change is happening in the direction of v. Note that we are identifying the point p with the vector p in $\mathbb{R}^n$, so that we can compute the term $p+tv$ legally. Unfortunately this trick only works for Euclidean space. If we want to generalise this to a smooth manifold M, then we would have to replace the "$p + tv$" term with a term that tells us how to evaluate f at points nearby p, in the direction of v. Fortunately, we have such an object: the flow! The map θ_t pushes p in the direction of v. This means that at least locally, we can reproduce the directional derivative of a smooth function, this time over M, by defining:

$$D_v f = \lim_{t\to 0} \frac{f(\theta_t(p)) - f(p)}{t}.$$

A quick computation shows that the above expression turns out the be equal to $v(f)$ evaluated at the point p. Indeed, we have

$$\begin{aligned}
\lim_{t\to 0} \frac{f \circ \theta_t(p) - f(p)}{t} &= \lim_{t\to 0} \frac{f \circ \gamma^p(t) - f(\gamma^p(0))}{t} \\
&= \frac{d}{dt}\bigg|_{t=0} f \circ \gamma^p \\
&= \frac{d(\gamma^p)^i(0)}{dt} \frac{\partial f}{\partial x^i} \\
&= v^i(p) \frac{\partial f}{\partial x^i}.
\end{aligned}$$

It follows that $D_v f(p) = v_p(f)$.

In Euclidean space we can also take directional derivatives of vector fields. The operation is rather similar to that above, except that now we evaluate a vector field w in $\mathfrak{X}(\mathbb{R}^n)$:

$$D_v w := \lim_{t\to 0} \frac{w(p+tv) - w(p)}{t} = \lim_{t\to 0} \frac{w_{p+tv} - w_p}{t}.$$

Naively, it might be fruitful to trade the "$p + tv$" terms for flows $\theta_t(p)$. However, if we do this we will face another problem: the vector $w_{\theta_t(p)}$ and w_p exist at different tangent spaces, and in a general manifold there is no way to subtract the two vectors. Instead, we need to somehow transfer the vector $w_{\theta_t(p)}$ from the tangent space $T_{\theta_t(p)}$ over to the tangent space T_pM. Fortunately, flows will come to save the day again. Recall that the flow θ_t is a diffeomorphism, with inverse given by θ_{-t}. Therefore we can transfer vectors using the pushforward of the inverse flow. The correct generalisation of the directional derivative of a vector field w in $\mathfrak{X}(M)$ is:

$$\lim_{t\to 0}\frac{(d\theta_{-t})_{\theta_t(p)}w_{\theta_t(p)} - w_p}{t}.$$

This expression is known as the *Lie derivative*, and is denoted by $\mathcal{L}_v w(p)$. One might wonder what this name has to do with the Lie theory that we have outlined thus far. As it turns out, there is a nice connection between the Lie derivative $(\mathcal{L}_v w)(p)$ and the commutator $[v, w]_p$. Let f be some smooth function of M. We can then consider how the Lie derivative acts on f:

$$\begin{aligned}
(L_v w)_p(f) &= \lim_{t\to 0}\frac{(d\theta_{-t})_{\theta_t(p)}w_{\theta_t(p)}(f) - w_p(f)}{t} \\
&= \lim_{t\to 0}\frac{(d\theta_{-t})_{\theta_t(p)}w_{\theta_t(p)}(f) - w_{\theta_t(p)}(f) + w_{\theta_t(p)}(f) - w_p(f)}{t} \\
&= \lim_{t\to 0}\frac{(d\theta_{-t})_{\theta_t(p)}w_{\theta_t(p)}(f) - w_{\theta_t(p)}(f)}{t} + \lim_{t\to 0}\frac{w_{\theta_t(p)}(f) - w_p(f)}{t} \\
&= \lim_{t\to 0}\frac{w_{\theta_t(p)}(f\circ\theta_{-t}) - w_{\theta_t(p)}(f)}{t} + \lim_{t\to 0}\frac{w_{\theta_t(p)}(f) - w_p(f)}{t} \\
&= \left(\lim_{t\to 0} w_{\theta_t(p)}\right)\left(\lim_{t\to 0}\frac{(f\circ\theta_{-t}) - f}{t}\right) + \lim_{t\to 0}\frac{w(f\circ\theta_t)(p) - w(f)(p)}{t} \\
&= w_p(-v(f)) + v_p(w(f) \\
&= v_p(wf) - w_p(vf) \\
&= [v, w]_p(f).
\end{aligned}$$

This gives us a nice geometric picture for commutator of vector fields: at a point, the commutator $[v, w]_p$ evaluates the directional derivative of the vector field w as it changes by an infinitesimal amount in the direction of v.

4 The Lie Algebra of a Lie Group

An algebra is a vector space equipped with a particular product operation. We have already seen this with our discussion of the tensor algebra $T(V)$, which is a vector space endowed with the tensor product $\otimes$, and the exterior algebra $\bigwedge(V)$, which is a vector space endowed with the wedge product $\wedge$. A Lie algebra is another type of algebra. The definition is as follows.

Definition 4.1. *A Lie algebra is a vector space* $\mathfrak{V}$ *equipped with a map* $[\cdot,\cdot] : \mathfrak{V} \times \mathfrak{V} \to \mathfrak{V}$ *satisfying the following properties for all vectors* u, v, w *in* $\mathfrak{V}$.

1. *Bilinearity:* $[au + bv, w] = a[u, w] + b[v, w]$,
2. *Antisymmetry:* $[v, w] = -[w, v]$,
3. *Jacobi Identity:* $[u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0$.

The bracket $[\cdot,\cdot]$ is commonly referred to as the *Lie bracket*, though it may also be called a commutator.[5]

Remark 4.2. *We have already secretly seen an example of a Lie algebra: a routine computation confirms that the commutator of vector fields is bilinear, antisymmetric, and satisfies the Jacobi identity. This means that the space of vector fields* $\mathfrak{X}(M)$ *equipped with the commutator bracket is a Lie algebra.*

Any Lie group G admits a unique Lie algebra, which is interpreted to be an infinitesimal approximation of the grouplike behaviour of G. We will denote by $\mathfrak{g}$ the Lie algebra corresponding to the Lie group G. Since every Lie group is a smooth manifold, we know what the Lie algebra should be – it should somehow be a space of tangent vectors of G. The natural prescription is to define $\mathfrak{g}$ to be equal to T_eG, the tangent space at the identity. We will now spell out the details of this prescription.

4.1 Left-Invariant Vector Fields

According to 4.2 the space of vector fields on a manifold has a natural commutator which turns it into a Lie algebra. In particular, this means that for

[5] The term "commutator" is really used for something of the form: $[X, Y] = XY - YX$, and it is not immediately clear that a Lie bracket always agrees with this expression. As it turns out, there is a useful result known as *Ado's Theorem* which says that any finite-dimensional Lie algebra can be realised as a subalgebra of some $\mathfrak{gl}(n)$. Since the Lie bracket of the general-linear Lie algebra is a commutator, and subalgebras are formed from the restriction of the Lie bracket of the parent space, we may indeed conclude that the Lie bracket of any finite-dimensional Lie algebra is indeed a commutator.

a Lie group G the space of vector fields $\mathfrak{X}(G)$ is a Lie algebra (once equipped with the natural commutator bracket). However, this Lie algebra is not *the* Lie algebra of G that we have motivated previously. The reason for this is that the space $\mathfrak{X}(G)$, viewed as a vector space over $\mathbb{R}$, is infinite dimensional. This conflicts with our prescription of $\mathfrak{g}$ as T_eG.

Although the space $\mathfrak{X}(G)$ is too large to be equal to the Lie algebra $\mathfrak{g}$, it is has a natural subspace that is finite-dimensional and isomorphic to $\mathfrak{g}$. This is the space of so-called *left-invariant* vector fields. We will define it properly after first making a few observations.

Recall that the group G admits a family of G-many diffeomorphisms L_g, which we called left translations. Since these are diffeomorphisms, we can use them to pushforward vector fields. We therefore say that a vector field V in $\mathfrak{X}(G)$ is *left-invariant* if

$$(L_g)_*v := (dL_g)v = v,$$

that is, if $(dL_g)_h(v_h) = v_{gh}$ for all g and h in G. We denote the space of all left-invariant vector fields of G by $\mathfrak{X}^{inv}(G)$. It should be clear that $\mathfrak{X}^{inv}(G)$ is a vector subspace of $\mathfrak{X}(G)$. Moreover, we have the following.

Proposition 4.3. *The space $\mathfrak{X}^{inv}(G)$ is a Lie subalgebra of $\mathfrak{X}$.*

Proof. It suffices to show that the space $\mathfrak{X}^{inv}(G)$ is closed under the commutator. So, suppose that v and w are left-invariant vector fields on G. We can use the fact that the pushforward of the commutator is the commutator of the pushforward (cf. 3.1) to conclude that

$$(L_g)_*[v, w] = [(L_g)_*v, (L_g)_*w] = [v, w],$$

as required. □

We will now relate the space $\mathfrak{X}^{inv}(M)$ to the Lie algebra $\mathfrak{g}$. Our first step is to again observe that the left translations are diffeomorphisms. This means that we can take their differential at the identity. This will be an isomorphism

$$(dL_g)_e : \mathfrak{g} \to T_gG.$$

Therefore, any fixed vector v in $\mathfrak{g}$ is mapped to a unique vector $(dL_g)_e(v)$ in T_gG. We can use this observation to define a vector field induced from v in $\mathfrak{g}$ by

$$v^L_g = v^L(g) := (dL_g)_e(v),$$

that is, v^L is the vector field that is obtained by pushing v in $\mathfrak{g}$ along all the left-translations. It turns out that this prescription induces a smooth vector field, and therefore v^L lies in $\mathfrak{X}(G)$.

Theorem 4.4. *The vector field v^L defined above is left-invariant. Moreover, the map from $\mathfrak{g}$ to $\mathfrak{X}^{inv}(G)$ that acts by $v \mapsto v^L$ is a bijection.*

Proof. To see the left-invariance we can simply compute the pushforward of v^L:

$$(L_g)_* v^L(h) = (dL_g)_{g^{-1}h} v^L(g^{-1}h) = (dL_g)_{g^{-1}h}(dL_{g^{-1}h})_e v = (dL_h)_e v = v^L(h).$$

To see that the map $v \mapsto v^L$ is bijective, we will show that it is injective and surjective. Suppose first that v and w are two elements of $\mathfrak{g}$. Then the difference $v - w$ is non-zero. This means that the pushforward along any L_g will be:

$$(L_g)_*(v - w) = (dL_g)_e(v - w) = (dL_g)_e(v) - (dL_g)_e(w).$$

Since $(dL_g)_e$ is a linear map, it sends 0 in T_eG to 0 in T_gG. It follows that $(dL_g)_e(v) - (dL_g)_e(w)$ is non-zero, and thus $(dL_g)_e(v) \neq (dL_g)_e(w)$. We may thus conclude that $v^L \neq w^L$. Hence the map $v \mapsto v^L$ is injective.

Suppose now that w is some element of $\mathfrak{X}^{inv}(G)$. We will show that there is some v in $\mathfrak{g}$ such that $w = v^L$. There is one obvious guess: we let v equal value of the vector field w evaluated at e. It follows quickly from the left-invariance of w that $w = w^L_e$. □

Using the above result, we can describe the Lie bracket on $\mathfrak{g}$ by transferring the commutator of left-invariant vector fields back to $\mathfrak{g}$ using the inverse map $v^L \mapsto v$. Indeed, suppose that we have two vectors v, w in $\mathfrak{g}$. We define the Lie bracket $[v, w]$ to be the unique vector in $\mathfrak{g}$ that satisfies

$$[v, w]^L = [v^L, w^L] := v^L \circ w^L - w^L \circ v^L.$$

Since the ^{L}-map is a bijection, the above is perfectly valid. Moreover, this bracket will be a Lie bracket since the commutator is.

4.2 The Exponential Map

According to our discussion thus far, we can always go from a Lie group to a Lie algebra by "differentiating", that is, by considering the tangent space at the identity T_eG. As it turns out, we can also go in the other direction and map T_eG to G. The map that does this is known as the *exponential map.*

4.2.1 One Parameter Subgroups

Before getting to the definition of the exponential map, we will digress into the integral curves of a Lie group. We start by making the following observation.

Lemma 4.5. *Every left-invariant vector field v on a Lie group G is complete.*

The gist of the proof is that we always have unique integral curves of v in $\mathfrak{X}^{inv}(G)$ passing through each point g in G. We can then use the left-translations to succesively extend the integral curves from a small domain $(-\epsilon, \epsilon)$ to the whole of $\mathbb{R}$.[6]

Let v be some member of $\mathfrak{X}^{inv}(G)$, and consider the global flow θ of v. The flow θ is a group action $\theta : \mathbb{R} \times G \to G$ defined by

$$\theta_t(g) = \gamma^g(t),$$

[6]This extension procedure is called the "uniform time lemma". See chapter 9 of Lee for the details.

that is, we flow along the integral curve passing though g for time t.

Suppose now that we have an integral curve γ of v starting at e in G. Moreover, fix some g that lies on γ, say at $g = \gamma(t)$. We can obtain "two" new integral curves from γ as follows.

1. L_g is a diffeomorphism, and diffeomorphisms send integral curves to integral curves (cf. 3.3). Since v is left-invariant, the map $L_g \circ \gamma$ is an integral curve of v that starts at $L_g \circ \gamma(0) = L_g(e) = g = \gamma(t)$.

2. We can use translations of the parameter t of γ to create new integral curves – we did this in Section 3.3, and denoted the translated curves by $\hat{\gamma}$. Consider the curve $\hat{\gamma} : \mathbb{R} \to G$ defined by $\hat{\gamma}(s) = \gamma(t+s)$. This is again an integral curve at v, which starts at $\hat{\gamma}(0) = \gamma(t+0) = \gamma(t) = g$.

Observe that the two integral curves described above are both integral curves of the vector field v with the same starting point g. Therefore, by the uniqueness of maximal integral curves, the two curves $L_g \circ \gamma$ and $\hat{\gamma}$ must be the same. Spelling this out, we have that

$$\gamma(t)\gamma(s) = g\gamma(s) = L_g\gamma(s) = \hat{\gamma}(s) = \gamma(t+s).$$

We may therefore conclude that the (maximal) integral curves of a left-invariant vector field are actually *group homomorphisms* from the additive group $(\mathbb{R}, +)$ into G. In the context of Lie groups, curves that are also group homomorphisms go by a special name.

Definition 4.6. *A one-parameter subgroup of a Lie group G is a group homomorphism $\gamma : \mathbb{R} \to G$.*

In our discussion above we have shown that the integral curves of a left-invariant vector field yield one-parameter subgroups. In fact, it can be shown that *all* the one-parameter subgroups of G can be obtained in this manner.

4.2.2 The Exponential Map Defined

We would like to create a map that relates the Lie algebra $\mathfrak{g}$ to the Lie group G. From our previous discussions, the space $\mathfrak{g}$ can either be seen as the tangent space T_eG, or as the space of left-invariant vector fields. According to the previous section, each of these left invariant vector fields generates a one-parameter subgroup of G. With this in mind, we can define the so-called *exponential map*, which sends elements of $\mathfrak{g}$ to elements of G. The map is defined by

$$\exp(v) = \gamma(1),$$

where γ is the one-parameter subgroup generated from the left-invariant vector field v^L. Equivalently, this can be seen as flowing e for unit time along the integral curve that is generated from v^L and starts at e.

There are several properties of the exponential map (see Chpt. 20 of Lee). For our purposes, we are most interested in the following.

Theorem 4.7. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Then the exponential map is a local diffeomorphism around 0 in $\mathfrak{g}$ and e in G.*