Fibre Bundles and Spin Structures

Part 2: Smooth Manifolds

David O'Connell

Okinawa Institute of Science and Technology
1919-1 Tancha, Onna-son, Okinawa 904-0495, Japan

1 Some Basics of Topology

A set X is a near-structureless collection of objects. Various fields of mathematics are defined in terms of endowing sets with some sort of additional structure. Some examples would be:

1. equipping X with a binary relation,
2. equipping X with a function $f : X \to X$,
3. equipping X with a binary operation $g : X \times X \to X$, or
4. fixing special designated elements of X.

Different types of structure emerge from stipulating different properties of the objects you are considering. A topology is precisely one of these structures. Formally, it is introduced as a designation of certain subsets of X to be *open*. This terminology comes from the open intervals (a, b) of the real line, and open sets are defined to mimic the behaviour of open intervals with respect to set-theoretic operations like unions and intersections. The intention is that this definition of topology is a generalisation of certain properties of continuous functions. The nasty ϵ/δ definition of a continuous function can be rephrased in topological terminology by using open subsets of sets of instead of open intervals of $\mathbb{R}$. Topology is introduced as a structure on a set that is preserved under continuous transformations.

In this formulation, topology is an extremely foundational and general notion. Given any set, there are many ways that one can define a topology. The set could consist of anything – points on a circle, a string of data, people in the room, and so on. The field of set-theoretic topology introduces and studies various properties that a topology may or may not have. To list but a few, there are many properties regarding:

- *countability*, that is, how many open sets are required to "build" the full topology,
- *compactness*, that is, whether the topology can run-off to infinity in some direction,
- *separability*, that is, whether points in the topological space can be separated by open sets, or functions, or whatever, and
- *connectedness*, that is, how many pieces of the topological space are there, and how well-connected they are.

Remark 1.1. *In this course we will not explicitly use many notions from general topology, so it is not worth spending too much time introducing them. In class I will introduce them when I need them, and in these notes I will casually make references to them. If there is any time that you do not recognise a concept from general topology, I can only refer you to Munkres' text on the subject, it's the one I use.*

2 Topological Manifolds

A topological manifold is a particular type of topological space that is locally equivalent to some fixed Euclidean space. In formal terminology, locality is expressed by open neighbourhoods, and equivalence is expressed by homeomorphisms. As such, the definition of a locally-Euclidean space is the following.

Definition 2.1. *A topological space X is called locally-Euclidean if for each point x in X, there exists some open neighbourhood U of X and a homeomorphism $\varphi : U \to V$ to some connected, open subset V of a Euclidean space $\mathbb{R}^n$.*

The requirement that the open subset V be connected is merely for convenience – it doesn't matter too much; since V is homeomorphic to U, without loss of generality we could always make V connected by considering the connected component U' of U that contains the point x. By the definition above, to every point in a locally-Euclidean space X we can associate an open chart U_x. The collection of all of these forms what is known as an *atlas*. We denote by $\mathcal{A}$ an atlas of X.

With this out of the way, we can now define what it means for a given topological space to be a topological manifold.

Definition 2.2. *A topological space X is called a topological manifold if the following conditions are met:*

1. *X is locally-Euclidean,*

2. *X is second-countable, and*

3. *X is Hausdorff.*

The requirement that a topological manifold be locally-Euclidean is obvious – this is precisely the intuition that we wanted to capture with our definition. However, the other two conditions are subtleties that should not go without justification. The requirement that a manifold be Hausdorff is in order to avoid certain unusual models which have "doubled" points that are superimposed on top of each other. The standard example is something like the line with two origins. The requirement that a topological manifold be second-countable is needed to ensure that a topological manifold is not "too big".[1] In order to proceed, we need to first fix some terminology.

Definition 2.3. *An atlas of a locally-Euclidean space X is a collection of charts (U, φ) that cover X.*

Before getting to the definition of a smooth manifold, we will first examine some of the possible consequences of the definition of topological manifolds. Suppose that we have a topological manifold M, and ket p be a point in M. Since M is locally Euclidean, there is some open neighbourhood U of p that is homeomorphic to some open neighbourhood of a fixed Euclidean space. But, in general there may well be multiple open neighbourhoods of our point p that satisfy this property. Suppose we have another chart (V, ψ) at p. We thus have something like that pictured in Figure 1. In such a situation, we can consider the function

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V).$$

This map is a homeomorphism between two open subsets of Euclidean space, and is known as a *transition function*. These functions will be pivotal in our definition of a smooth manifold.

In general there are usually several ways to construct the same manifold. Take for instance the manifold S^1. This can be constructed,

a) as a subspace of $\mathbb{R}^2$,

[1]Moreover, there are several results regarding the topological properties of manifolds that would not hold if the manifold were not Hausdorff or second-countable. See Chapter 1 of Lee for more details.

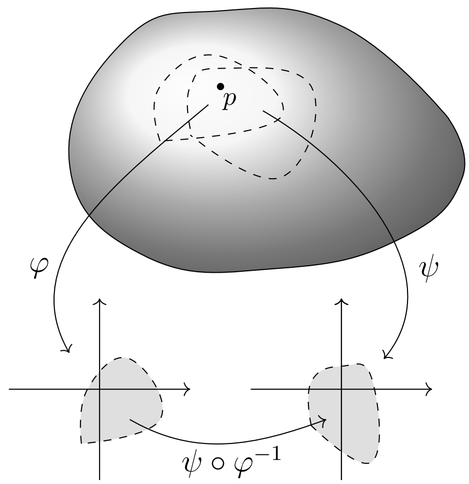

Figure 1: The idea behind the transition functions. We take the intersection of two charts, and use this to induce a homeomorphism between two open subsets of Euclidean space.

b) by gluing two unit intervals along their boundaries,

c) as a quotient space of $\mathbb{R}/\mathbb{Z}$, or

d) by gluing together two or more open intervals together.

In some way or another, all of these constructions can be generalised into constructions for other manifolds. Type (a) is an *extrinsic* construction, namely one that appeals to the properties of some ambient, higher dimensional Euclidean space to be defined. Type (b) is a type of surgery, where products of different dimensional disks and spheres are removed to create an artificial boundary, which can then be glued along. Type (c) involves quotienting a pre-existing manifold by a group action of a particular type. Type (d) is something called an *intrinsic* construction, and will be the center of attention for the rest of this section.

In the modern formulation of differential geometry we define our manifolds intrinsically. You may read this as a matter of convention that turns out to be just fine.[2] It turns out that such a construction is always possible, since we can decompose the atlas into a disjoint union of subsets of Euclidean space, and then we can glue them together using the transition maps. The formal statement is as follows.

Theorem 2.4. *Suppose that U_α is a collection of open subsets of $\mathbb{R}^n$, and $\varphi_{\alpha\beta} : U_{\alpha\beta} \to U_{\beta\alpha}$ is a homeomorphism such that*

1. $\varphi_{\alpha\beta}^{-1} = \varphi_{\beta\alpha}$
2. $\varphi_{\alpha\alpha} = id_{U_\alpha}$
3. $\varphi_{\alpha\beta} \circ \varphi_{\beta\gamma} \circ \varphi_{\gamma\alpha} = id.$

Then the relation $\sim$ which relates $(p, \alpha) \sim (q, \beta)$ iff $\varphi_{\alpha\beta}(p) = q$ is a well-defined equivalence relation on the disjoint union $\bigsqcup U_\alpha$. Moreover, the resulting quotient space is a topological manifold.

In a casual sense, the above theorem is simply saying that a manifold is really only a collection of charts and a collection of functions that can be used to glue the charts together nicely.

3 Smooth Manifolds

A bare topological manifold does not have enough structure to be able to meaningfully talk about calculus. We can remedy this by endowing a topological manifold with something known as a *smooth structure*, which formalises the idea that local patches of Euclidean space are pasted together smoothly. We start by considering two overlapping charts.

[2] There is a fairly practical reason for this, coming from GR. Since we want to model our universe as a manifold, and we exist within it, and the universe is all that there is, we might want a theoretical framework that allows us to determine features of the manifold from the "inside" without appealing to something extra-universal.

Definition 3.1. *Let M be a topological manifold, with (U, φ) and (V, ψ) two charts with non-empty intersection. We say that the charts (U, φ) and (V, ψ) are smoothly compatible if their transitions function $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ are smooth in the Euclidean sense.*

If two charts have empty intersection then we will also regard them as smoothly compatible. In the previous section we defined an atlas to be a collection of charts that cover M. As an extension of this idea, we have the following definition.

Definition 3.2. *An atlas $\mathcal{A}$ is called smooth if it consists of pairwise smoothly-compatible charts.*

One might expect that a smooth manifold is a topological manifold equipped with a smooth atlas. This is not quite right – as a matter of convenience we will take this smooth atlas to be maximal.

Definition 3.3. *A smooth structure on M is a maximal smooth atlas.*

We will justify the maximality of a smooth structure in the next section when we discuss smooth real-valued functions. For now, we can finish this section with the precise definition of a smooth manifold.

Definition 3.4. *A smooth manifold is a topological manifold M equipped with a smooth structure.*

With this definition in mind, we now start to lift the ideas of calculus into the manifold setting. In the remainder of this document we will assume that M is a smooth manifold.

3.1 Smooth Functions

We can use the differentiable structure of a smooth manifold to specify when a continuous map between manifolds is smooth. Essentially this is done by appealing to the smoothness of the local Euclidean patches, and then by arguing that the definition is independent of the particular charts chosen to witness it. We will start with the simpler case of a function from M to $\mathbb{R}$.

Definition 3.5. *A map $f : M \to \mathbb{R}$ is called smooth at a point p in M if there exists a chart (U, φ) at p such that the composition $f \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}$ is smooth in the Euclidean sense.*

We can extend the terminology by saying that a function $f : M \to \mathbb{R}$ is smooth if it is smooth at every point p. We will also define

$$C^\infty(M) := \{f : m \to \mathbb{R} \mid f \text{ is smooth}\}.$$

As a small remark: it turns out that the existence of a single chart is enough to guarantee that *every* chart yields a smooth coordinate representation of the function f. We will discuss the more-general version of this idea in a moment.

3.1.1 $C^\infty(M)$ as an Algebra

We can take the collection of all real-valued smooth functions $f : M \to \mathbb{R}$ and study what structure it has. The key idea to understanding the structure of $C^\infty(M)$ is to appeal to the rich structure of the real line itself. For any two smooth functions f and g in $C^\infty(M)$ we can define the following:

- an addition operation, $(f + g) : M \to \mathbb{R}$ defined by $(f + g)(p) = f(p) +_{\mathbb{R}} g(p)$,
- a product operation, $(f \cdot g) : M \to \mathbb{R}$ defined by $(f \cdot g)(p) = f(p) \cdot_{\mathbb{R}} g(p)$,
- an additive unit, $0_{C^\infty(M)} : M \to \mathbb{R}$ defined by sending all of M into $0 \in \mathbb{R}$, and finally
- a multiplicative unit, $1_{C^\infty(M)} : M \to \mathbb{R}$ defined by sending all of M into $1 \in \mathbb{R}$.

One can easily confirm that these maps are actually smooth, since addition, multiplication and constant functions are smooth in the Euclidean setting. As one might expect, the set $C^\infty(M)$ together with these operations does forms an algebra – the various properties follow from the ring properties of $\mathbb{R}$.

3.2 Smooth Maps Between Manifolds

We can generalise the notion of a smooth function to a smooth map between manifolds by appealing to coordinate representations in the domain and the image. The definition is as follows.

Definition 3.6. *A function $f : M \to N$ between smooth manifolds M and N is called smooth at a point p in M if there exists some chart (U, φ) at p in M and a chart (V, ψ) at $f(p)$ in N such that the map $\psi \circ f \circ \varphi^{-1}$ is smooth in the Euclidean sense. We say that f is a smooth map if it is smooth at every point p in M.*

Although the definition of a smooth map only requires the existence of a single chart, in fact it can be shown that *every* coordinate representation of a smooth map is a smooth map between Euclidean spaces. The full argument is a little involved, so we will just provide a partial justification.

Suppose that $f : M \to N$ is smooth, and consider a point p in M. By definition there are charts $(U_\alpha, \varphi_\alpha)$ at p and (V_α, ψ_α) at $f(p)$ such that the coordinate representation $\psi_\alpha \circ f \circ \varphi_\alpha^{-1}$ is smooth. Now, suppose that (U_β, φ_β) and (V_β, ψ_β) are different charts at p and $f(p)$ respectively. We would like to show that the coordinate representation $\psi_\beta \circ f \circ \varphi_\beta^{-1}$ is smooth. In order to do this, we will cleverly represent this function as a composition of functions that we know to be smooth. As a quick shorthand, let us write $U_{\alpha\beta}$ for the intersetion $U_\alpha \cap U_\beta$, and similarly for the V's. We then have the following.

$$\begin{aligned}
\psi_\beta \circ f \circ \varphi_\beta^{-1} &= \psi_\beta \circ id_{V_{\alpha\beta}} \circ f \circ id_{U_{\alpha\beta}} \circ \varphi_\beta^{-1} \\
&= \psi_\beta \circ (\psi_\alpha^{-1} \circ \psi_\alpha) \circ f \circ (\varphi_\alpha^{-1} \circ \varphi_\alpha) \circ \varphi_\beta^{-1} \\
&= (\psi_\beta \circ \psi_\alpha^{-1}) \circ (\psi_\alpha \circ f \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \varphi_\beta^{-1}).
\end{aligned}$$

The first and third functions are smooth since we are working with smooth manifolds, and the central function is smooth by assumption. Since we have a composition of smooth functions, we can see that the coordinate representation of f in terms of the β-chart is indeed smooth as well.

4 Tangent Spaces

We have seen that by endowing locally-Euclidean topological spaces with a smooth structure, it is possible to define a meaningful notion of smooth functions between manifolds. We will now show that this buys us even more – by appealing to the smooth structure in relation to the real line, we can define linear-algebraic structures such as vectors, dual vectors and general tensors. In this section we will discuss vectors on manifolds.

Geometrically, a vector space consists of a collection of arrows, that is, things with a magnitude and a direction. In more abstract terms, a vector space consists of a collection of objects that can be added and scaled in an intuitive way. This abstract view of vector spaces is actually quite useful – it allows us to construct vector spaces out of many other things that don't geometrically look like arrows. We have already seen this in play – in Section 3 we argued that the space $C^\infty(M)$ can be seen as a ring or as an infinite-dimensional algebra (i.e. a vector space).

There are several equivalent ways in which to define the tangent spaces of a smooth manifold. These depend on which perspective you like – if you have a preference for

1. *geometry*, then you may see tangent space at a point p as the infinitesimal directions in which you can traverse by starting at p, and if you have a preference for

2. *algebra*, then you may see the tangent space at p as a space of certain derivative operators acting on $C^\infty(M)$ at the point p.

In either case, we will obtain a real-valued vector space of dimension the same as the manifold. The geometric picture should be clear – the space of directions at a point should locally look like the space of directions of n-dimensional Euclidean space. As for the algebra of derivations, these should be able to be expressed in some coordinate form as the directional derivatives at a point in $\mathbb{R}^n$.

4.1 Equivalence Classes of Curves

Suppose we have a smooth curve $\gamma : \mathbb{R} \to M$, defined in such a way that the curve passes through the point p in M at $t = 0$, as in Figure 2. By assumption γ is smooth, so we can use a chart to form a local representation of the curve as the function $\varphi \circ \gamma : \mathbb{R} \to \mathbb{R}^n$. This is now a curve in some local patch of Euclidean space, so it can be differentiated. We define the velocity vector of γ to be

$$v_\gamma := \frac{d}{dt}\Big|_{t=0} (\varphi \circ \gamma)(t). \tag{1}$$

We can then try and associate a tangent vector to each γ passing through p. However, there is a slight subtlety here – in general there could be multiple curves with the same such derivative. So, we will actually take an equivalence class of curves, where we define two curves to be equivalent if their derivatives agree on a fixed coordinate representation. We can then define the tangent space T_pM to be the collection of all curves passing through p, quotiented by the equivalence relation that identifies curves with equal velocity vectors.

4.2 Derivations of $C^\infty(M)$ at a Point

Another construction of the tangent space T_pM uses the concept of a derivation. A derivation is a generalised derivative-type operator that can be defined on any algebra. Since we are working with the specific algebra $C^\infty(M)$, we will define derivations as follows.

Definition 4.1. *A derivation of $C^\infty(M)$ at the point p is a function $v : C^\infty(M) \to \mathbb{R}$ such that*

1. *v is a linear map, meaning that $v(f + g) = v(f) + v(g)$ and $v(\lambda f) = \lambda v(f)$, and*

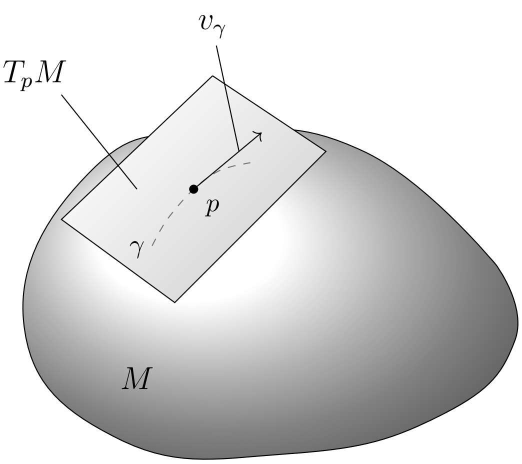

Figure 2: The tangent space T_pM, viewed as equivalence classes of curves.

2. v satisfies the Leibniz rule: $v(fg) = v(f)g(p) + f(p)v(g)$.

It turns out that the collection of these derivations has a clear linear structure – we can add use the vector space structure of $\mathbb{R}$ to define all of the required structure.[3]

4.3 The Differential of a Smooth Map

Differentiation can be seen as an infinitesimal linearization of a function. Therefore, if we were to try and differentiate (in some sense of the word) a smooth map $f : M \to N$, we would expect there to be some "lifting" of the map f into the tangent spaces of the respective manifolds, as pictured in Figure 3. This is indeed what happens, and the map in question is called the *differential* of the map f. For now we will only discuss this differential at a point, but the global version will be used later on in the course.

[3]The addition of two derivations is: $(v + w)(f) = v(f) +_{\mathbb{R}} w(f)$, and we can define scalar multiplication by $(\lambda v)(f) := v(\lambda f)$, and finally we can define a "zero" derivation v_0 by asserting that $v_0(f) = 0$ for all $f \in C^\infty(M)$.

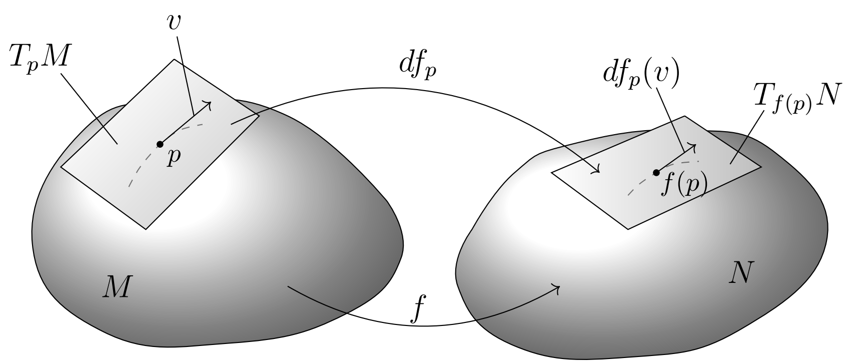

Figure 3: The differential of f at p.

Depending on the particular construction of the tangent space, the specifics of the map $df_p : T_pM \to T_{f(p)}N$ differ. Indeed – if we take a geometric view, then df_p will need to act on (equivalence classes) of curves, and if we take an algebraic view, then df_p will act on derivations of $C^\infty(M)$.

- Let γ be a curve passing through the point p at $t = 0$. We can take the composition $f \circ \gamma$, which will be a curve in N that passes through the point $f(p)$ at $t = 0$. Therefore, we may define df_p to be the map which sends the equivalence class v_γ to the equivalence class $v_{f\circ\gamma}$.

- Let v be a derivation of $C^\infty(M)$ at p, i.e. v sends functions of $C^\infty(M)$ to $\mathbb{R}$. For any function g in $C^\infty(N)$, the composition $g \circ f$ is a function in $C^\infty(M)$. Hence we can define $df_p(v) : C^\infty(N) \to \mathbb{R}$ to be the function that acts by $g \mapsto v(g \circ f)$. One can confirm that $df_p(v)$ is actually a derivation of $C^\infty(N)$ at $f(p)$.

Independent of the choice of the construction, we always have the following nice properties of the pointwise differential of a smooth map.

Theorem 4.2. *Let $f : M \to N$ be a smooth map, and let $df_p : T_pM \to T_{f(p)}$ be the differential of the function f at the point p in M.*

1. *df_p is a linear map.*

2. *If f is a diffeomorphism, then df_p is an isomorphism of vector spaces.*

4.4 The Coordinate Basis of the Tangent Space

In a smooth manifold a local chart (U, φ) is a diffeomorphism. According to the previous theorem, this causes the differential $d\varphi_p$ to act as an isomorphism between the vector spaces T_pM and $T_0\mathbb{R}^n$. This means that we can transfer the standard basis of $T_0\mathbb{R}^n$ over to T_pM using the differential of φ^{-1}. The idea follows from the observation that φ can be decomposed into n-many smaller functions x^i:

$$\varphi(p) = \begin{pmatrix} x^1(p) \\ x^2(p) \\ \vdots \\ x^n(p) \end{pmatrix},$$

where x^i can be seen as functions in $C^\infty(U)$. We can interpret these x^i as linearly-independent coordinates in $\mathbb{R}^n$, which we can then use to create a basis of $T_0\mathbb{R}^n$. So, let $\frac{\partial}{\partial x^i}$ be the canonical basis of the tangent space $T_0\mathbb{R}^n$ – the basis of partial derivatives. We can then define

$$\left.\frac{\partial}{\partial x^i}\right|_p := d(\varphi^{-1})_0 \left(\left.\frac{\partial}{\partial x^i}\right|_0 \right) = (d\varphi_p)^{-1} \left(\left.\frac{\partial}{\partial x^i}\right|_0 \right).$$

This is usually called a coordinate basis of the tangent space T_pM. This means that any vector v in T_pM can be represented as a linear combination

$$v = v^i \left(\left.\frac{\partial}{\partial x^i}\right|_p \right),$$

where we have assumed the Einstein summation convention.[4]

4.4.1 Transformation Properties of Vectors

We will now study the situation when there are overlapping charts (U, φ) and (V, ψ) at a point p. We denote the coordinate functions of φ by x^i, and

[4]Note that not every basis needs to be a coordinate basis. In principal, we can transform a coordinate basis under a transformation M that preserves linearity and dimension, and this will give another non-coordinate basis. Secretly, all of these transformations form a Lie group known as the General Linear group.

the coordinate functions of ψ by $\tilde{x}^i$. We can use the local expression of the differential to create the transformation properties of a vectors components. We are going to consider the differential of the transition function $\psi \circ \varphi^{-1}$. Since the transition function is a map between Euclidean spaces, we first need to consider the coordinate expression of the differential of a function.

Let's start with the easier case of a smooth function F between Euclidean spaces, say $\mathbb{R}^m$ with coordinates $x^=i$ and $\mathbb{R}^n$ with coordinates y^j. The differential at a point r in $\mathbb{R}^m$ will be a map $dF_r : T_r\mathbb{R}^m \to T_{f(r)}\mathbb{R}^n$. We can study this map by studying it's behaviour on basis vectors. We have:

$$dF_r\left(\left.\frac{\partial}{\partial x^i}\right|_r\right)(g) = \left.\frac{\partial}{\partial x^i}\right|_r(g \circ F) = \frac{\partial g}{\partial y^j}(F(r))\frac{\partial F^j}{\partial x^i}(r) = \left(\frac{\partial F^j}{\partial x^i}(r)\left.\frac{\partial}{\partial y^j}\right|_{F(r)}\right)g$$

for any smooth function g in $C^\infty(\mathbb{R}^m)$. Note the application of the multidimensional chain rule in the second equality. Since the above holds for any smooth g, we may conclude that

$$dF_r\left(\left.\frac{\partial}{\partial x^i}\right|_r\right) = \frac{\partial F^j}{\partial x^i}(r)\left.\frac{\partial}{\partial y^j}\right|_{F(r)}. \tag{2}$$

This means that the basis vectors are transformed according to the Jacobian of the function.

Suppose now that $f : M \to N$ is a smooth map, and consider two charts (U, φ) at p and (V, ψ) at $f(p)$. The coordinate representation of df_p will be the differential of the coordinate representation of f, that is, we need to compute the function

$$d(\psi \circ f \circ \varphi^{-1})_{\varphi(p)} : \mathbb{R}^m \to \mathbb{R}^n.$$

Using the coordinate functions as basis elements, we can evaluate

$$d(\psi \circ f \circ \varphi^{-1})_{\varphi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right).$$

Since this is a map between tangents of Euclidean spaces, we can just use formula 2 with $F = \psi \circ f \circ \varphi^{-1}$ and $r = \varphi(p)$. The end result is:

$$d(\psi \circ f \circ \varphi^{-1})_{\varphi(p)} \left(\frac{\partial}{\partial x^i}\bigg|_{\varphi(p)} \right) = \frac{\partial(\psi \circ f \circ \varphi^{-1})^j}{\partial x^i}(\varphi(p)) \frac{\partial}{\partial y^j}\bigg|_{\psi \circ f(p)}. \tag{3}$$

We can use this rather unattractive expression in a simpler form to obtain the transformation properties of a tangent vector in T_pM. Suppose we have two charts (U, φ) and (V, ψ) at p. By the previous section, there are two coordinate bases of T_pM, say $\{\frac{\partial}{\partial x^i}\}$ for the chart (U, φ) and $\{\frac{\partial}{\partial \tilde{x}^i}\}$ for the chart (V, ψ). The transition function $\psi \circ \varphi^{-1}$ tells us how to go from x-coordinates to $\tilde{x}$-coordinates – that is, $\psi \circ \varphi^{-1}(x) = \tilde{x}^i(x)$. Here we are cheating slightly by identifying the coordinate function x^i with a point in $\varphi(U \cap V)$, but that's ok. We can plug all of this data into equation (3) by taking $N = M$ and f equal to the identity. This yields:

$$d(\psi \circ \varphi^{-1})_{\varphi(p)} \left(\frac{\partial}{\partial \tilde{x}^i}\bigg|_{\varphi(p)} \right) = \frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p)) \frac{\partial}{\partial \tilde{x}^j}\bigg|_{\psi(p)}.$$

Using a clever insertion of the identity function, we can obtain the following transformation law for the coordinate basis of T_pM.

$$\begin{aligned}
\frac{\partial}{\partial x^i}\bigg|_p &= d(\varphi^{-1})_{\varphi(p)} \left(\frac{\partial}{\partial x^i}\bigg|_0 \right) \\
&= d(\psi^{-1} \circ \psi \circ \varphi^{-1})_{\varphi(p)} \left(\frac{\partial}{\partial x^i}\bigg|_0 \right) \\
&= d(\psi^{-1})_{\psi(p)} \circ d(\psi \circ \varphi^{-1})_{\varphi(p)} \left(\frac{\partial}{\partial x^i}\bigg|_0 \right) \\
&= d(\psi^{-1})_{\psi(p)} \left(\frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p)) \frac{\partial}{\partial \tilde{x}^j}\bigg|_{\psi(p)} \right) \\
&= \frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p)) d(\psi^{-1})_{\psi(p)} \left(\frac{\partial}{\partial \tilde{x}^j}\bigg|_{\psi(p)} \right) \\
&= \frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p)) \left(\frac{\partial}{\partial \tilde{x}^j}\bigg|_p \right).
\end{aligned}$$

The above string of equivalences tells us how the basis vectors transform during a change of coordinates. As for the components of arbitrary vectors, suppose we have $v = v^i \frac{\partial}{\partial x^i}$ in (U, φ)-coordinates, and we have $v = \tilde{v}^i \frac{\partial}{\partial \tilde{x}^j}$ in (V, ψ)-coordinates. Then:

$$v^i \frac{\partial}{\partial x^i}\bigg|_p = v^i \left(\frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p)) \left(\frac{\partial}{\partial \tilde{x}^j}\bigg|_p \right) \right) = \left(\frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p)) v^i \right) \frac{\partial}{\partial \tilde{x}^j}\bigg|_p.$$

It follows that

$$\tilde{v}^i = \frac{\partial \tilde{x}^i}{\partial x^j}(\varphi(p)) v^j. \tag{4}$$

5 The Cotangent Space

We can use the tangent space to create another vector space associated to points in the manifold, these are the so-called *cotangent* spaces. The "co" here refers to the duality of the two vector spaces. We will start by describing the cotangent space in terms of it's coordinate basis, and then we will observe the transformation properties of the components of a covector. All of the pieces are already in place – we will heavily rely on the differential again, and we will use results detailed in section 4 where necessary.

5.1 The Dual of a Tangent Space

For any vector space V, we may always form the dual vector space V^* as follows:

$$V^* := \{\omega : V \to \mathbb{R} \mid \omega \text{ is a linear map}\},$$

with addition and scalar multiplication defined pointwise. The resulting structure is indeed a vector space, and it can be shown to be isomorphic to V. If e_i is a basis for V, then we can define a basis ϵ^i of V^* by requiring that

$$\epsilon^j(e_i) = \delta_i^j,$$

where ∂_i^j is the standard Kronecker delta symbol.[5] Such a basis goes by the name of a *dual basis*.

We can apply this *-construction to the tangent spaces T_pM of a smooth manifold to obtain a collection of dual vector spaces T_p^*M, known as *co*tangent spaces. Elements of the cotangent space are often called dual vectors or covectors. We will use both terms interchangeably.

5.2 The Coordinate Basis of T_p^*M

We saw in Section 4.4 that the data of a local chart (U, φ) secretly encodes a collection of n-many functions in $C^\infty(M)$ known as the coordinate functions, which we denoted by x^i. These were used in conjunction with the differential to define the coordinate basis $\{\frac{\partial}{\partial x^i}|_p\}$ of the tangent space T_pM. We will now construct a similar story for the cotangent space T_p^*M. The underlying idea is that we want to construct a *dual* basis ϵ^i, which acts in a particular way on the basis of T_pM:

$$\epsilon^i\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \delta_j^i.$$

The goal of this section is to derive an explicit description of the ϵ^i.

Our first step is to make the following observation. Recall that the differential of a smooth map $f : M \to N$ is a map which sends tangent vectors in T_pM to tangent vectors in $T_{f(p)}N$. There is no harm in applying this in the case that f is a smooth function. In this setting, the differential df_p will be a map from T_pM to $T_{f(p)}\mathbb{R}$, where here we are viewing $\mathbb{R}$ as a smooth manifold. However the tangent space at a point in $\mathbb{R}$ is equal to $\mathbb{R}$ itself, so we can meaningfully write

$$df_p : T_pM \to \mathbb{R}$$

[5] Secretly, this Kronecker symbol encodes a dot product which allows the lowering and raising of indices.

for any function f in $C^\infty(M)$. This particular case of the differential will act on any vector v in T_pM by $df_p(v) = v(f)$. This is linear, since it is a differential. As such, we can say that the differential df_p of a smooth function f is actually a covector.

Since df_p is in T_p^*M, we can write it as a linear expression of the basis ϵ^i, that is $df_p = A_i\epsilon^i$, where A_i are components that can be determined by:

$$A_i = A_j\delta_i^j = A_j\epsilon^j\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = df_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \left.\frac{\partial}{\partial x^j}\right|_p(f) = \frac{\partial f}{\partial x^j}(p).$$

This means that for any smooth function f, the differential df_p can be expressed as

$$df_p = \frac{\partial f}{\partial x^i}(p)\epsilon^i.$$

Consider now the coordinate functions x^i. Technically these are smooth functions in $C^\infty(U)$, but the reasoning above still applies. Using the above expression, we can the differential of the x^i as

$$dx_p^i = \frac{\partial x^i}{\partial x^j}(p)\epsilon^j = \delta_j^i\epsilon^j = \epsilon^i.$$

Thus we have determined the value of the dual basis of T_p^*M – the cotangent space is spanned by the differentials of the coordinate functions.

5.2.1 Transformation Properties for Covectors

The transformation properties of a covector are easily to obtain. We can use the transformation properties of tangent vectors, together with the properties of the dual basis to deduce that:

$$\begin{aligned}
\omega_i &= \omega\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) \\
&= \omega\left(\frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p))\left(\left.\frac{\partial}{\partial \tilde{x}^j}\right|_p\right)\right) \\
&= \frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p))\omega\left(\left.\frac{\partial}{\partial \tilde{x}^j}\right|_p\right) \\
&= \frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p))\omega_j.
\end{aligned}$$