Fibre Bundles and Spin Structures

Part 1: Introduction

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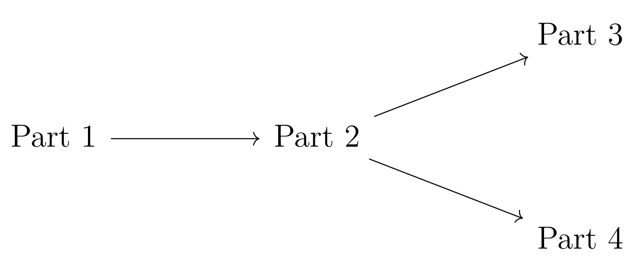
Overall, this course will introduce the subject of differential geometry. The idea is to set you up with enough of the required material for tackling more complicated subjects in mathematics and physics that you might want to study later in your PhD – stuff like gauge theories and spin geometry.

There are four parts to this course. These are:

1. Manifolds and Lie Theory (Week 1-4)
2. Fibre Bundles (Week 5/6)
3. Connections and Curvature (Week 7/8)
4. Spin Structures (Week 9/10)

The dependency is below.



A Summary of Ideas

Throughout the term we will be introducing lots of new terminology and ideas. So, the following is a rough summary of most of the ideas we will consider in this course, mostly written in chronological order.

1. **Manifolds.** These are generalisations of curves and surfaces to higher dimensions, and are defined "intrinsically", which means without assuming that the manifold is embedded into some Euclidean space. Manifolds are made of patches of Euclidean space, which usually means that one can define a differential structure, allowing the techniques of calculus to be passed into this more-general setting.

2. **Vectors and Tensors**. We can use the differential structure of a manifolds to construct tangent spaces. This is done by differentiating curves passing through a point. The machinery of linear algebra can then be used to define covectors and tensors in a natural way.

3. **Differential Forms**. These are a special type of antisymmetric tensor. They are important for two reasons: they can be integrated, and they admit a special derivative operator (called the exterior derivative).

4. **de Rham Cohomology**. This is an algorithm that turns differential forms and their exterior derivative into a collection of algebraic objects. It turns out that this algorithm can be used to test the topology of the manifold.

5. **Lie Theory**. Lie groups are manifolds which admit a compatible group structure. Using the differential structure of the manifold, one can create an infinitesimal linear approximation to the group, known as the Lie algebra. Information about the Lie algebra can be used to discern information about the Lie group, and vice-versa.

6. **Actions and Representations**. Symmetries are a collection of transformations which leave an object unchanged. Therefore groups are a

structured collection of transformations. An action is formed by allowing the transformations in a group to act on some other object. A representation is a special type of action in which we let a Lie group act on a vector space.

7. **The Tangent Bundle**. We can take the collection of all tangent spaces to all points in a manifold, and endow this with a manifold structure of its own. The collection of all these tangent spaces are thus "bundled" together into something known as the tangent bundle.

8. **Vector Bundles**. We can also assign other vector spaces to each point in a manifold, and turn the collection of these into a bundle of its own. The general object is called a vector bundle. Locally, a vector bundle looks like a Cartesian product, but globally it may be "twisted".

9. **Principal Bundles**. These are like vector bundles, except that the data associated to each point is now a set carrying an action of a fixed Lie group.

10. **Associated Bundles and Frame Bundles**. Vector bundles and principal bundles are two sides of the same coin. From a vector bundle, we can form a principal bundle (called the frame bundle), and from a principal bundle and a group representation we can form a vector bundle (called an associated bundle).

11. **Connections**. A connection is an extra piece of structure that allows us to relate bundle data associated to different (but nearby) points in a manifold. This is achieved by transporting the data from one place to another. There are different versions of connections for different types of bundles. Connections can be defined as a specification of a "horizontal" direction, or can be equivalently seen as a matrix of one-forms which send vectors into elements of a Lie algebra.

12. **Curvature**. We can apply a type of exterior derivative to a connection to yield an expression of curvature, which is a two-form. Geometrically,

this curvature measures the effect of transporting data around a small loop.

13. **Characteristic Classes**. Since curvature is a type of differential form, we can use it in integrals. Some of these integrals end up mapping into de Rham cohomology classes. Sometimes there are special classes that only depend on topological structure. These are called "characteristic" because the are independent on the choice of connection/curvature.

14. **Čech Cohomology**. This is a type of cohomology that combinatorially tracks the way that local charts of a manifold intersect each other. It can be used to find obstructions to the existence of certain types of bundles.

15. **Clifford Algebras**. The space of all differential forms on a vector space comprises an algebra called the exterior algebra. A Clifford algebra uses a metric to "deform" the exterior algebra of the vector space. An important subgroup of the Clifford algebra is the spin group, which can be shown to be a universal covering of the special orthogonal group (which are the symmetries of the vector space/metric combo).

16. **Spinors**. The spin group is a Lie group, so it has a representation theory. A space of spinors is a vector space that carries a particular type of representation of a spin group.

17. **Spin Structures**. We can use the spin group to define a principal bundle over a manifold. This bundle might not exist, and the existence can be determined by Čech cohomology.

Symbols Used (Subject to Updates)

X,Y,Z	Sets
$f, f', ...$	Functions
$\circ$	Function composition
τ	Topology
$U, U', ...$	Open sets
G	(Lie) group
$\cdot$	Group operation
$\bullet$	Group action
$\mathfrak{g}$	Lie algebra corresponding to G
$[\cdot, \cdot]$	Lie bracket
γ	Curve
M, N	Smooth Manifold
g	Metric (tensor)
V, W	Vector space
v, w	Vector (field)
$\Omega^p(M)$	p-forms over M
ω	Differential form
V^*	Vector space dual to V
T_pM	Tangent space of M at the point p
d	Exterior derivative
$\mathcal{L}_v$	Lie derivative along vector field v
∇_v	Covariant derivative along v
(E, π_E, M)	Vector Bundle E over M
s	Section
(P, π, M)	Principal G-bundle over M
∇	Ehresmann connection
$\mathcal{H}$	Horizontal Distribution
A	Connection one-form
F	Curvature form
D	Covariant Exterior Derivative

Further Reading

Of course none of the content I will be presenting is my original work. In preparing for this course I made use of several texts, since different authors use a different level of mathematical rigour. In future lecture notes I will try and be as honest/precise as possible with which sources I used. For now, here is a non-exhaustive list of resources for the various topics listed above.

- *Introduction to Smooth Manifolds* by John Lee
- *Geometry, Topology and Physics* by Mikio Nakahara
- *Topology, Geometry and Gauge Fields* by John Naber (both books)
- *Differential Geometry* by Loring W. Tu
- *Differential Geometry* by Clifford Henry Taubes
- *Spin Geometry* by Michelson/Lawson

There is a lot of overlap between these books, so I would suggest trying each of them out and seeing which one works for you.