Topological recursion in the F-world

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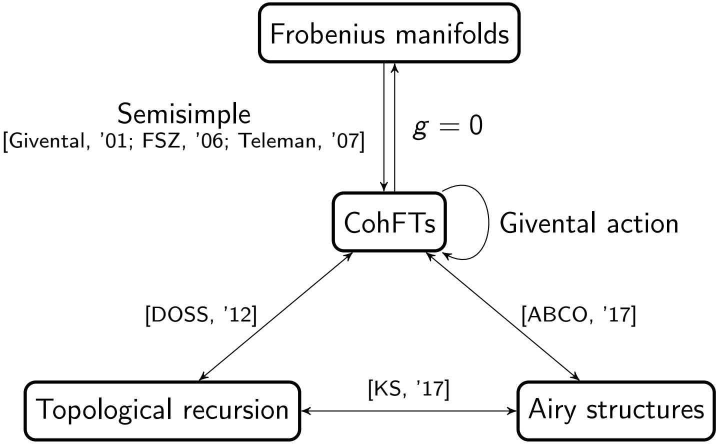
"Classical" picture



Flat F-manifolds

Definition [Hertling-Manin, '98]

A flat F-manifold $(M, \nabla, \cdot, e)$ is the data of a manifold M with

- a connection ∇ in the tangent bundle TM;
- an algebra structure $(T_pM, \cdot_p)$ with unit on each tangent space;
- additional axioms.

Locally there exist analytic functions $F^\alpha(t^1, \dots, t^N)$, $1 \leq \alpha \leq N$, such that the second derivatives

$$c^\gamma_{\alpha\beta} = \partial_{t^\alpha}\partial_{t^\beta}F^\gamma$$

are the structure constants of the algebra. $\underline{F} = (F^1, \dots, F^N)$ is called vector potential of the flat F-manifold.

F-Cohomological field theories

Definition [Buryak-Rossi, '18]

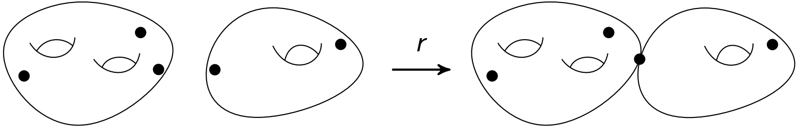
An F-CohFT is the data of a vector space V and a collection of linear maps

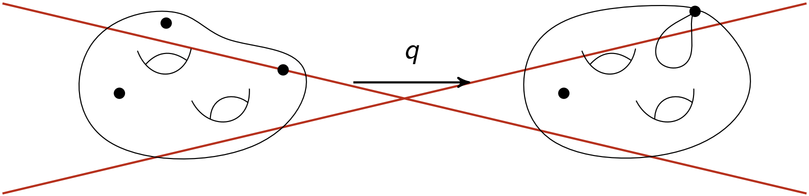
$$\Omega_{g,1+n} \colon V^* \otimes V^{\otimes n} \longrightarrow H^\bullet(\overline{\mathcal{M}}_{g,1+n})$$

indexed by integers $g, n \geq 0$ such that $2g - 2 + (1 + n) > 0$, satisfying the following axioms.

- $\Omega_{g,1+n}$ is equivariant for the action of the symmetric group $\mathbb{S}_n$ permuting simultaneously the tensor factors of $V^{\otimes n}$ and the last n marked points in $\overline{\mathcal{M}}_{g,1+n}$.
- Ω is compatible with the gluing map (of separating kind).

F-Cohomological field theories





From F-CohFTs to F-manifolds

Given an F-CohFT $(\Omega_{g,1+n})_{g,n\geq 0}$, then the functions

$$F^{\alpha}(\underline{t}) = \sum_{n\geq 2} \frac{1}{n!} \sum_{1\leq \alpha_1,\dots,\alpha_n \leq N} \left(\int_{\overline{\mathcal{M}}_{0,1+n}} \Omega_{0,1+n}(e^{\alpha} \otimes \bigotimes_{i=1}^{n} e_{\alpha_i}) \right) \prod_{i=1}^{n} t^{\alpha_i}$$

form a vector potential for a flat F-manifold.

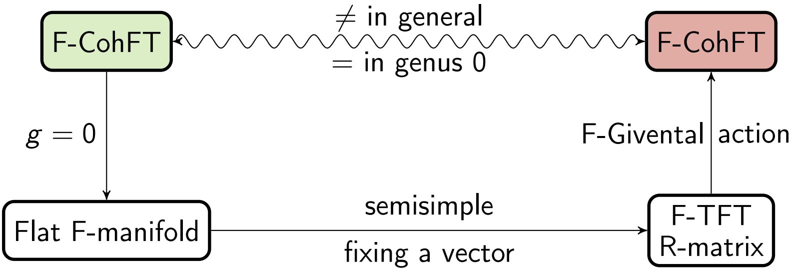
But is it true also the other way around?

A reconstruction of an F-CohFT amounts to the data of its $\deg 0$ part, i.e. an F-TFT $(V, \cdot, w)$, and the action of an R-matrix $R \in \mathrm{End}(V)[[z]]$. This generalises the usual Givental action defined for CohFTs.

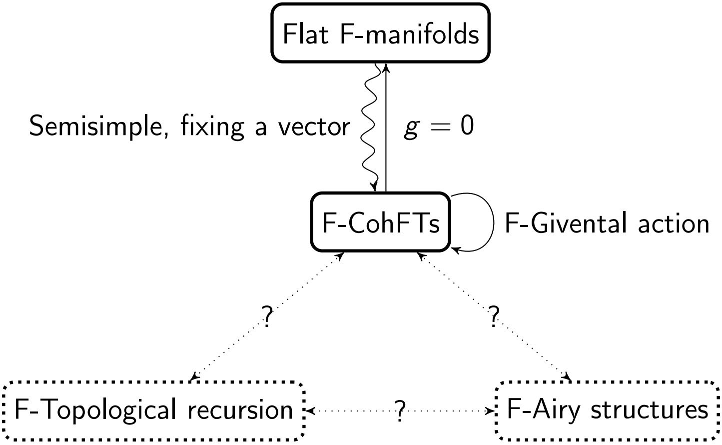
From F-manifolds to F-CohFTs

Theorem [Arsie-Buryak-Lorenzoni-Rossi, '20]

Given a flat F-manifold semisimple at the origin together with a vector in its tangent space, one can uniquely define an F-CohFT.



F-picture


Flat F-manifolds
Semisimple, fixing a vector
$g = 0$
F-CohFTs
F-Givental action
?
?
F-Topological recursion
?
F-Airy structures

F-Airy structures

Definition

An F-Airy structure on a vector space V is the data of tensors $A \in \mathrm{Hom}(\mathrm{Sym}^2 V, V)$, $B \in \mathrm{Hom}(V^{\otimes 2}, V)$, $C^\circ \in \mathrm{Hom}(V, V^{\otimes 2})$, $C^\bullet \in \mathrm{Hom}(\mathrm{Sym}^2 V, V)$, $D \in V$.

We define F-TR amplitudes

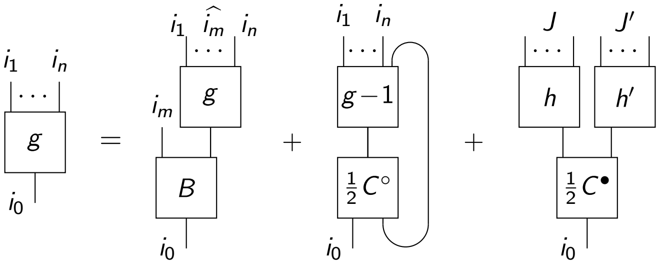
$$F_{g,1+n} \in \mathrm{Hom}(\mathrm{Sym}^n V, V)$$

by induction on $2g - 2 + (1+n) > 0$ with integers $g, n \geq 0$. Set $F_{0,3} = A$, $F_{1,1} = D$. Taken a basis $(e_i)_{i \in I}$ of V, we'll write

$$F_{g,1+n}(e_{i_1} \otimes \cdots \otimes e_{i_n}) = F^{i_0}_{g;i_1,\ldots,i_n} \, e_{i_0}$$

F-Airy structures

$$F^{i_0}_{g;i_1,\dots,i_n} = \sum_{m=1}^{n} B^{i_0}_{i_m,a}\, F^{a}_{g;i_1,\dots,\widehat{i_m},\dots,i_n} + \frac{1}{2}\, C^{\circ\, i_0,k}_{a}\, F^{a}_{g-1;i_1,\dots,i_n,k} + \frac{1}{2}\, C^{\bullet\, i_0}_{a,b} \sum_{\substack{h+h'=g \\ J \sqcup J' = \{i_1,\dots,i_n\}}} F^{a}_{h;J}\, F^{b}_{h';J'}\,.$$



F-Topological field theories

Definition

An F-topological field theory (F-TFT) is the data $(V, \cdot, w)$ of a commutative, associative algebra $(V, \cdot)$ together with an element $w \in V$. To an F-TFT one can associate the amplitudes

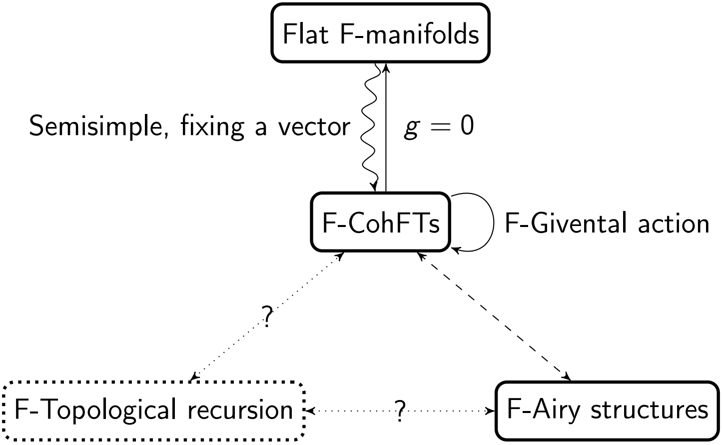
$$\mathcal{F}_{g,1+n} \colon v_1 \otimes \cdots \otimes v_n \longmapsto v_1 \cdots v_n \, w^g$$

Theorem

$\left(A = B = C^{\bullet} \colon (v_1, v_2) \longmapsto v_1 \cdot v_2, C^{\circ} \colon v \longmapsto v \otimes w, D = \frac{1}{2} w\right)$ define an F-Airy structure on V and the amplitudes of the associated F-TFT can be computed by F-TR.

We can generalise the action on F-CohFT to define a similar F-Givental group action onto the amplitudes of an F-Airy structure.

F-picture 2.0



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F-spectral curve

Definition

An F-spectral curve is a quintuple $(\mathcal{C}, x, y, \omega_{0,2}^{\circ}, \omega_{0,2}^{\bullet})$ s.t.

- $\mathcal{C}$ is a smooth complex curve equipped with two meromorphic functions x and y, with finitely many simple zeroes of $\mathrm{d}x$;
- $\omega_{0,2}^{\circ}$ and $\omega_{0,2}^{\bullet}$ are two fundamental bidifferentials of the second kind on $\mathcal{C}^2$.

We define the maps

$$\mathcal{P}_{-}^{\star} : \phi(z) \longmapsto \sum_{\alpha \in \mathfrak{a}} \operatorname{Res}_{w=\alpha} \phi(w) \Big(\int_{\alpha}^{z} \omega_{0,2}^{\star}(\cdot, w) \Big)$$

We set $\omega_{0,1} = y\mathrm{d}x$ and introduce the recursion kernels

$$K_{\alpha}^{\star}(z_0|z) = \frac{1}{2} \frac{\int_{\sigma_\alpha(z)}^{z} \omega_{0,2}^{\star}(\cdot, z_0)}{\omega_{0,1}(z) - \omega_{0,1}(\sigma_\alpha(z))}.$$

F-TR à la Eynard-Orantin

Then we can build two sequences $(\omega^{\bullet}_{g,1+n}, \omega^{\circ}_{g,1+n} = (\mathcal{P}^{\circ}_{-})^{\otimes n}\omega^{\bullet}_{g,1+n})$

$$\omega^{\bullet}_{g,1+n}(z_0|z_1,\ldots,z_n) =$$

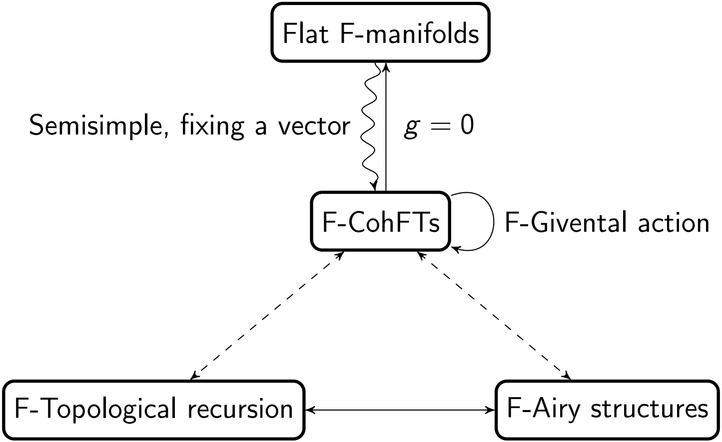
$$= \sum_{\alpha\in\mathfrak{a}} \operatorname*{Res}_{z=\alpha} K^{\bullet}_{\alpha}(z_0|z)\Bigg(\mathcal{P}^{\circ}_{-}\otimes\mathcal{P}^{\circ}_{-}[\omega^{\bullet}_{g-1,2+n}](z|\sigma_\alpha(z),z_1,\ldots,z_n)$$

$$+ \sum_{\substack{h+h'=g\\ J\sqcup J'=\{z_1,\ldots,z_n\}}}^{\text{no }(0,1)} \omega^{\bullet}_{h,1+|J|}(z|J)\,\omega^{\bullet}_{h',1+|J'|}(\sigma_\alpha(z)|J')\Bigg)$$

Theorem

Expanding on two natural bases of differentials, the coefficients of these two sequences coincide. Moreover they encode the F-TR amplitudes of an F-Airy structure.

F-picture final form



F-Thank you!

References

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