GENERAL RELATIVITY MIDTERM EXAM

Exercise 1. *Linearized GR is formulated in terms of a symmetric tensor field $h_{\mu\nu}$ in flat spacetime (secretly, this is a perturbation of the flat metric). The action reads:*

$$S = S_{matter} + \frac{1}{2}\int d^4x\, h_{\mu\nu}T^{\mu\nu} + S_{gravity}\ , \tag{1}$$

where S_{matter} is the action of matter without the effect of gravity, $T^{\mu\nu}$ is the matter's stress-energy tensor, and $S_{gravity}$ is the action of $h_{\mu\nu}$ without the effect of matter:

$$S_{gravity} = -\frac{1}{64\pi G}\int d^4x \left(\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho} - 2\partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho} + 2\partial_\mu h^{\mu\nu}\partial_\nu h^\rho_\rho - \partial_\mu h^\rho_\rho \partial^\mu h^\sigma_\sigma\right)\ . \tag{2}$$

Derive from the action (1) *the field equations for $h_{\mu\nu}$ in the presence of gravitational sources $T^{\mu\nu}$. Show that these equations require energy-momentum conservation $\partial_\mu T^{\mu\nu} = 0$. (As discussed in the lecture, this implies the need for a non-linear completion of the theory.)*

Exercise 2. *Consider Euclidean 3d space, in Cartesian coordinates $x^a = (x, y, z)$ and spherical coordinates $\tilde{x}^a = (r, \theta, \phi)$. At each point, consider three bases for the tangent space: (i) the usual Cartesian basis $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$, (ii) the coordinate basis associated with $\tilde{x}^a$, and (iii) the basis $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}})$ of unit vectors along the r, θ and ϕ directions. Write down the metric g (i.e. its matrix elements) in each of these bases. Find the transformation matrix A between each pair of bases, and verify the transformation law for the metric.*