GENERAL RELATIVITY MIDTERM EXAM

Exercise 1. *Linearized GR is formulated in terms of a symmetric tensor field $h_{\mu\nu}$ in flat spacetime (secretly, this is a perturbation of the flat metric). The action reads:*

$$S = S_{matter} + \frac{1}{2}\int d^4x\, h_{\mu\nu}T^{\mu\nu} + S_{gravity}\ , \tag{1}$$

where S_{matter} is the action of matter without the effect of gravity, $T^{\mu\nu}$ is the matter's stress-energy tensor, and $S_{gravity}$ is the action of $h_{\mu\nu}$ without the effect of matter:

$$S_{gravity} = -\frac{1}{64\pi G}\int d^4x\left(\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho} - 2\partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho} + 2\partial_\mu h^{\mu\nu}\partial_\nu h^\rho_\rho - \partial_\mu h^\rho_\rho\partial^\mu h^\sigma_\sigma\right)\ . \tag{2}$$

Derive from the action (1) *the field equations for $h_{\mu\nu}$ in the presence of gravitational sources $T^{\mu\nu}$. Show that these equations require energy-momentum conservation $\partial_\mu T^{\mu\nu} = 0$. (As discussed in the lecture, this implies the need for a non-linear completion of the theory.)*

Exercise 2. *Consider an exponentially expanding universe, with coordinates* (t, x, y, z)*, and metric given by:*

$$g_{\mu\nu}(t) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & e^{2Ht} & 0 & 0 \\ 0 & 0 & e^{2Ht} & 0 \\ 0 & 0 & 0 & e^{2Ht} \end{pmatrix}\ . \tag{3}$$

H is a constant, known as the Hubble constant. The motion of particles in this spacetime (just like in flat spacetime) is governed by an action proportional to the worldline's length:

$$S = -m\int\sqrt{-g_{\mu\nu}(t)dx^\mu dx^\nu}\ . \tag{4}$$

1. *Write this action in a "non-relativistic" form $S = \int L dt$, and derive the equations of motion for the particle's spatial coordinates $x^i(t)$. In particular, identify a conserved momentum p_i.*

2. *Consider a particle that starts at $x^i = 0$ at time $t = 0$, with initial velocity $v_{(0)}$ in the x direction. A remarkable feature of the spacetime* (3) *is that the particle will never reach beyond a certain position $x = x_{final}$. Find x_{final} as a function of $v_{(0)}$, assuming that $v_{(0)}$ is much smaller than the speed of light.*

3. Find x_{final} in the opposite limit, of a particle traveling at the speed of light. As in Week 3's homework, you may achieve this limit by sending $m \to 0$ while keeping the momentum p_i finite.