Black hole spacetimes and Symmetries

Lecture #9

$(-,+,+,\dots)$

In the last lecture we looked at conserved charges generating diffeomorphisms on spacetime — importantly, we saw that they correspond to only surface terms. And we looked at the specific example of GR.

In the next few lectures, we will continue to focus on GR, specifically black holes. Today, we will try to collect some of the main features of black hole spacetimes, with standard examples, especially those which will be needed in the upcoming lectures.

Subject of black holes is extremely vast — here, will try to convey the main ideas, especially those needed later.

Throughout the course, we have stressed on the importance of boundaries — boundary conditions, boundary action, surface terms etc.

An important notion in GR is that of <u>asymptotic</u> boundaries — boundaries that are "infinitely" far away (in a sense to be specified below).

In particular, <u>asymptotically flat</u> spacetimes are of obvious interest:

asymptotically flat — "approaches Minkowski @ infinity".

Recall, from some lectures back:

asymptotic fall-off condition for a scalar field on flat background

$$\phi = \mathcal{O}\left(\frac{1}{r^{(D-2)/2}}\right)$$

@ large r

First: "put into box", IR regulate

put $\partial\Sigma$ (originally @ spacelike infinity $r = \sqrt{\delta_{ij}x^i x^j} \to \infty$)

@ finite $r = R$

Then : take $\lim R \to \infty$.

Such asymptotic b.c restrict behaviour of fields in certain regions @ infinity

But in GR : Metric is a dynamical field

→ don't have a background (flat) metric to define fall-off conditions on curvature etc.

→ also, in general, no global inertial reference frame to consider limits like $r \to \infty$

One possibility : look for <u>any</u> coord. system (x^μ) s.t. $g_{\mu\nu}$ behave as needed @ large coord. values

e.g. $g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(1/r)$ @ large r.

But, need to eventually verify coord.

④ invariance of results — cumbersome, challenging

→ the precise method of actually taking a limit in a meaningful coord-ind. way is not always guaranteed.

Thus, seek:
- intrinsic, coord-ind., notion of "infinities" (asymptotic boundaries)
- construction where "infinity" is brought to a finite distance, so that can instead consider b.c. like

$$\phi\Big|_{I_0} = \mathcal{O}\left(\frac{1}{r^{(D-2)/2}}\right)$$

↑ "spatial infinity"

Results in : Asymptotically flat
definition of

→ Return in next lecture (BMS group of symmetries, etc.)

→ Here : enough to think "like flat near infinity"

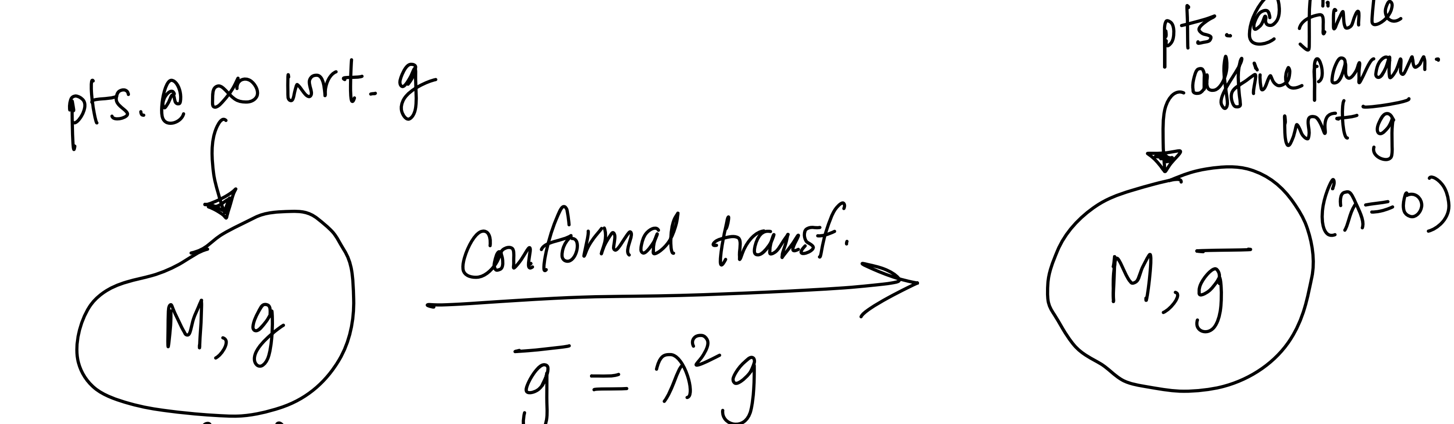
Remark : in general, consider asymptotic flatness @ spatial and null infinities only (i.e. $\lim R \to \infty$ in spacelike and null directions),

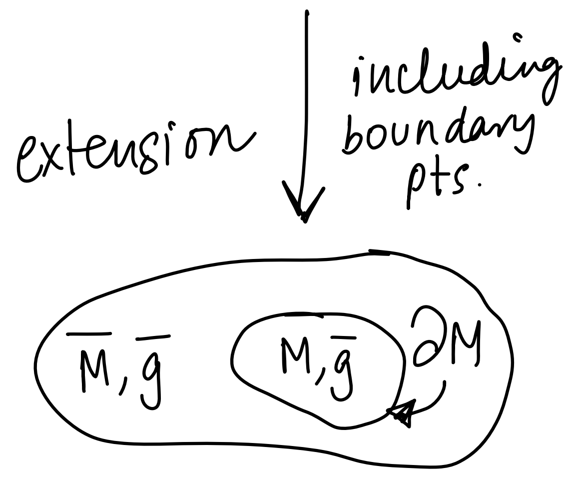
but <u>not</u> @ timelike infinities (i.e. $t \to \pm\infty$ @ fixed R) — because, we may be interested in physical systems where matter (e.g. a star) is present @ early and late times, so metric cannot approach flat.

In all this,

an imp. tool: <u>Conformal Compactification</u>

(infinite boundaries are brought to a finite distance in a larger auxilliary spacetime)





Choose λ s.t. $(M, \bar{g})$ is extendible to some $(\bar{M}, \bar{g})$.

Aside: Not all conformal transformations are diffeomorphisms. A diffeo Ξ s.t. $\Xi^* g = \lambda^2 g$ is called a conformal isometry.

Example: Minkowski $(M, \eta_{\mu\nu})$

$$ds^2 = -dt^2 + dr^2 + r^2 \underbrace{d\Omega^2}$$

not (pre)symp. 2-form !

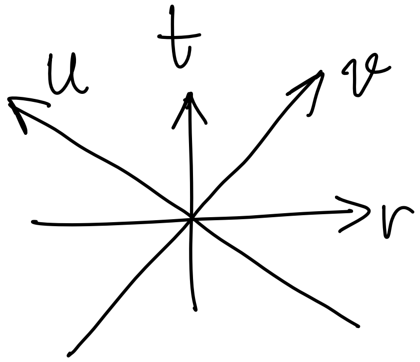
angular line element (S^2) $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$

not (pre)symplectic potential !

Retarded, advanced time coords:

$u = t - r$

$v = t + r$



$r \geq 0$

$$ds^2 = -du\,dv + \frac{1}{4}(u-v)^2 d\Omega^2$$

$-\infty < u, v < \infty$

New coords: p, q

$u = \tan p$

$v = \tan q$

$-\pi/2 < p \leq q < \pi/2$

$$g \equiv ds^2 = \frac{1}{4\cos^2 p \cos^2 q}\left[-4\,dp\,dq + \sin^2(q-p)\,d\Omega^2\right]$$

"Infinity": $r \to \infty$, $|t| \to \infty$ original coords

$|p| \to \pi/2$, $|q| \to \pi/2$ new coords

$\lambda := 2\cos p \cos q$

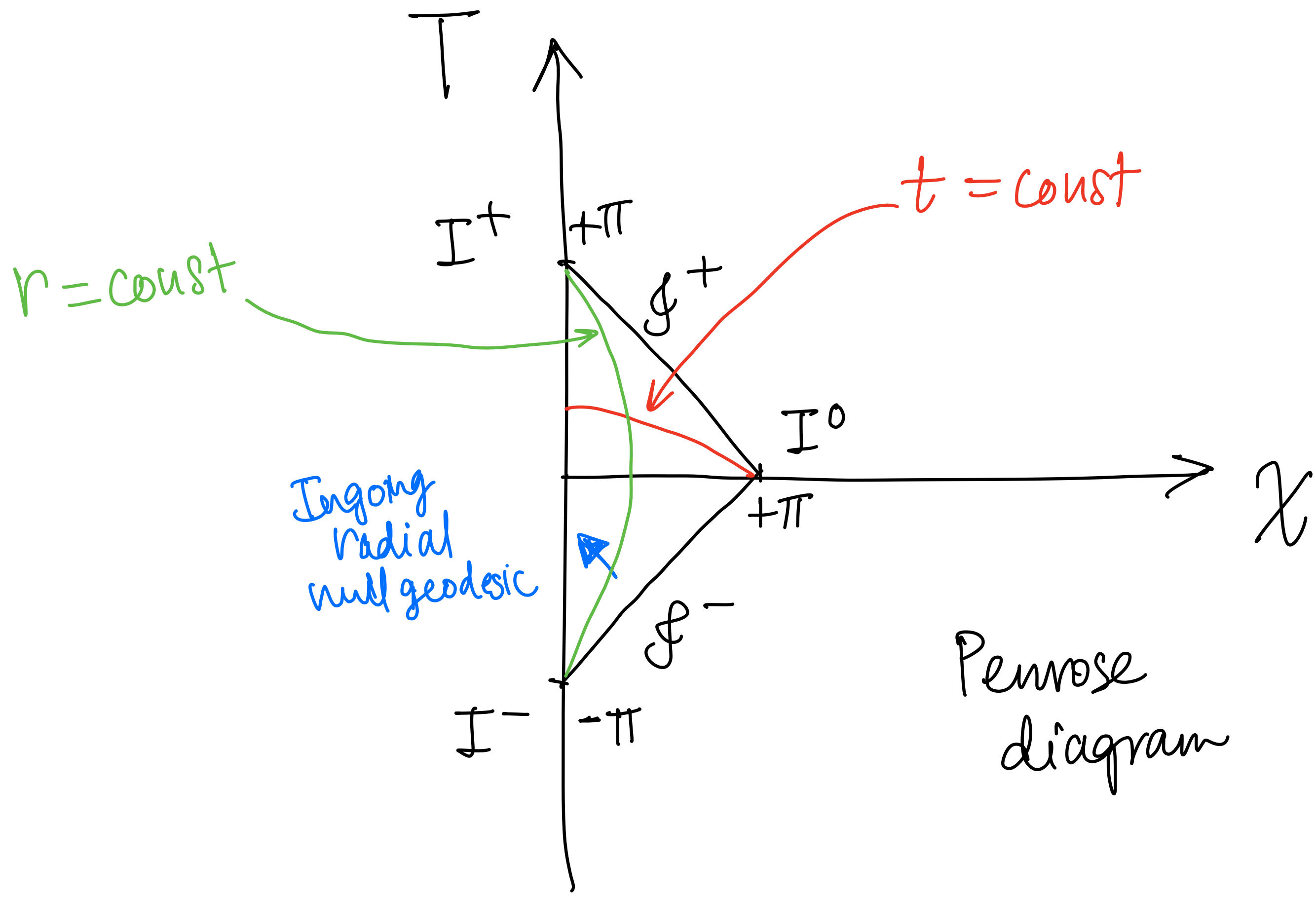
$$\bar{g} = \lambda^2 g = -4\,dp\,dq + \sin^2(q-p)\,d\Omega^2$$

Finally: $T = q + p \in (-\pi, \pi)$

$\chi = q - p \in [0, \pi)$

$$\bar{g} = -dT^2 + d\chi^2 + \sin^2\chi\, d\Omega^2$$

(of course: flat metric is conformally flat)



Conformal infinities:

$\mathscr{I}^+$ future null infty $(r \to \infty, t \to \infty)$: Region toward which outgoing null rays extend

$\mathscr{I}^-$ past null infty $(r \to \infty, t \to -\infty)$: Region from which ingoing null rays come

I^0 spacelike infty : toward which spacelike
$(r \to \infty, \text{ finite } t)$ geodesics extend

I^+ future timelike infty : toward which timelike
$(t \to \infty, \text{ finite } r)$ geodesics extend

I^- past timelike infty : from which timelike
$(t \to -\infty, \text{ finite } r)$ geodesics come

Here: $(\bar{M}, \bar{g})$: Einstein static universe
$\mathbb{R} \times S^3$, $T \in (-\infty, \infty)$
$\chi \in [0, \pi]$

$(M, \bar{g})$: Penrose diagram alone

(M, g) : Original Minkowski

boundary ∂M in $\bar{M}$ $\longleftrightarrow$ infinities in Minkowski

Symmetries

Isometry: diffeo $F_\xi \in \mathrm{Diff}(M)$

s.t. $\mathcal{L}_\xi g = 0$.

vector field $\xi \in \mathcal{X}(M)$

diffeo generator

$$\mathcal{L}_\xi g = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$$

$\Longrightarrow$ for isometry:

$$\nabla_{(\mu} \xi_{\nu)} = 0 \qquad \text{Killing's eqn.}$$

Killing Vector Field (KVF)

i.e. KVFs are generators of 1-parameter group of isometries

Stationary: An asymptotically flat spacetime is stationary if it admits a KVF k that is asymptotically timelike.

(notion of asymptotic time translation symmetry, in generic curved M).

Stationary $\Rightarrow$ $\exists$ coords. (x^μ) s.t. $g_{\mu\nu}$ fns. are independent of coord. time

Static: Stationary + Time reversal symmetry (i.e. invariance under $t \to -t$)

Axisymmetry: An asymp. flat spacetime is axisymmetric if it admits a KVF m that is asymptotically spacelike and generates isometries isomorphic to $U(1)$ (i.e. all orbits of m are closed).

Stationary and Axisymm:

Asymp. flat spacetime which is stationary, axisymm. and $[k, m] = 0$.

Spherically symmetric: if its isometry group contains an $SO(3)$ subgroup with orbits that are 2-spheres.

In fact, there is only a very limited family of asymp. flat, stationary $\underbrace{\text{BH}}$ solns. in GR

↳ by BH spacetime, here we broadly mean: ∃ an event horizon s.t. spacetime is non-singular on and outside it.

→ several uniqueness results

- Schwarzschild : unique spherically symm. vacuum soln.

Birkhoff's thm : Any spherically symm. vacuum soln. of Einstein's eqns. is static and isometric to Schwarzschild.

1-parameter family : mass M

Related, Israel's thm: Any asymp. flat static vacuum BH soln. is spherically symm. and isometric to Schwarzschild.

- Reissner–Nordström: unique asymp. flat spherically symm. soln. of Einstein – Maxwell eqns.

2-parameter family: M, electric charge $\mathbb{E}$
(excluding non-phys. magnetic charge)

- Kerr: unique asymp. flat, stationary and axisymm. vacuum soln.

[Carter–Robinson thm.]

2-parameter family: M, angular momentum $\mathbb{J}$

Together:

- Kerr-Newman family of solns:

  asymp-flat, stationary, axisymm

  BH soln. to Einstein-Maxwell

Parameters : M, $\mathbb{J}$, $\mathbb{E}$

Charged, rotating BHs

Asymp-flat, BH spacetimes:

Schw.	RN	Kerr	KN
vacuum, spherically symm (⇒ static)	Einstein-Maxwell, spherically symm, static	vacuum, stationary, axisymm	Einstein-Maxwell, stationary, axisymm.
Parameter M	M, $\mathbb{E}$	M, $\mathbb{J}$	M, $\mathbb{J}$, $\mathbb{E}$

Schwarzschild BH

Schw. coords. (t, r, θ, φ)

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

- Regular for $r > 2M$
- Static ($\because$ all $g_{\mu\nu} = g_{\mu\nu}(x_i)$ and ds^2 inv. under $t \to -t$)
- KVF $k = \partial_t$

$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$
(spherical symm.)

By Birkhoff's thm: spacetime exterior to ___any___ sph. symm. body is described by Schw. Interior then depends on the details of $T_{\mu\nu}$ inside. Here, we consider interior $r<2M$ to also be empty/vacuum ($\because$ want to extend to all r and study the soln.....)

- Singularity @ $r = 2M$ ← Coordinate singularity (see below)

  @ $r = 0$ ← Curvature (hence, physical) singularity
  $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \propto M^2/r^6$

⇒ Coords. (t, r, θ, φ) valid for $0 < r < 2M$, $r > 2M$ separately

→ Need extension, to cover all $r > 0$.

- Eddington-Finkelstein (EF) extension

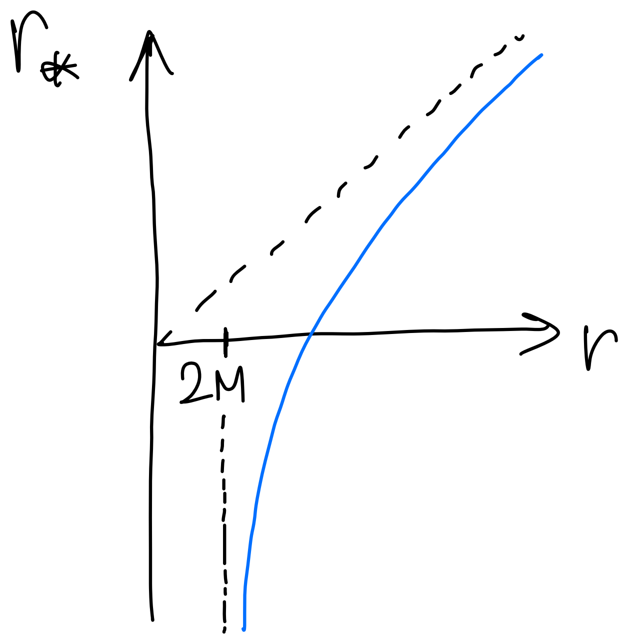
We see above that cannot go from exterior to interior smoothly since g_{rr} diverges @ $r = 2M$.

→ want to access $r<2M$ in a non-singular way

Regge-Wheeler radial coord. ("tortoise" coord.)

$$dr_* = \frac{dr}{\left(1-\frac{2M}{r}\right)}$$

$$r_* = r + 2M \ln\left|\frac{r-2M}{2M}\right|$$



Advanced, Retarded null coords:

$$v = t + r_*$$
$$u = t - r_*$$

Radial Geodesic eqns. give:

outgoing

$$\frac{dt}{dr} = \pm\left(1-\frac{2M}{r}\right)^{-1}$$

ingoing

Ingoing EF coords: (v, r, θ, φ)

$dv = 0$, $v = \text{const}$: ingoing radial null geodesics

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2\,dv\,dr + r^2 d\Omega^2$$

$\Rightarrow$ no singularity @ $r = 2M$.

But, $r = 2M$: special surface
one-way membrane
"event horizon"

No causal curve in $r < 2M$ can reach ∞ on exterior $(\mathscr{I}^+)$.

In other words, once cross $r = 2M$, cannot come back out to $r > 2M$.

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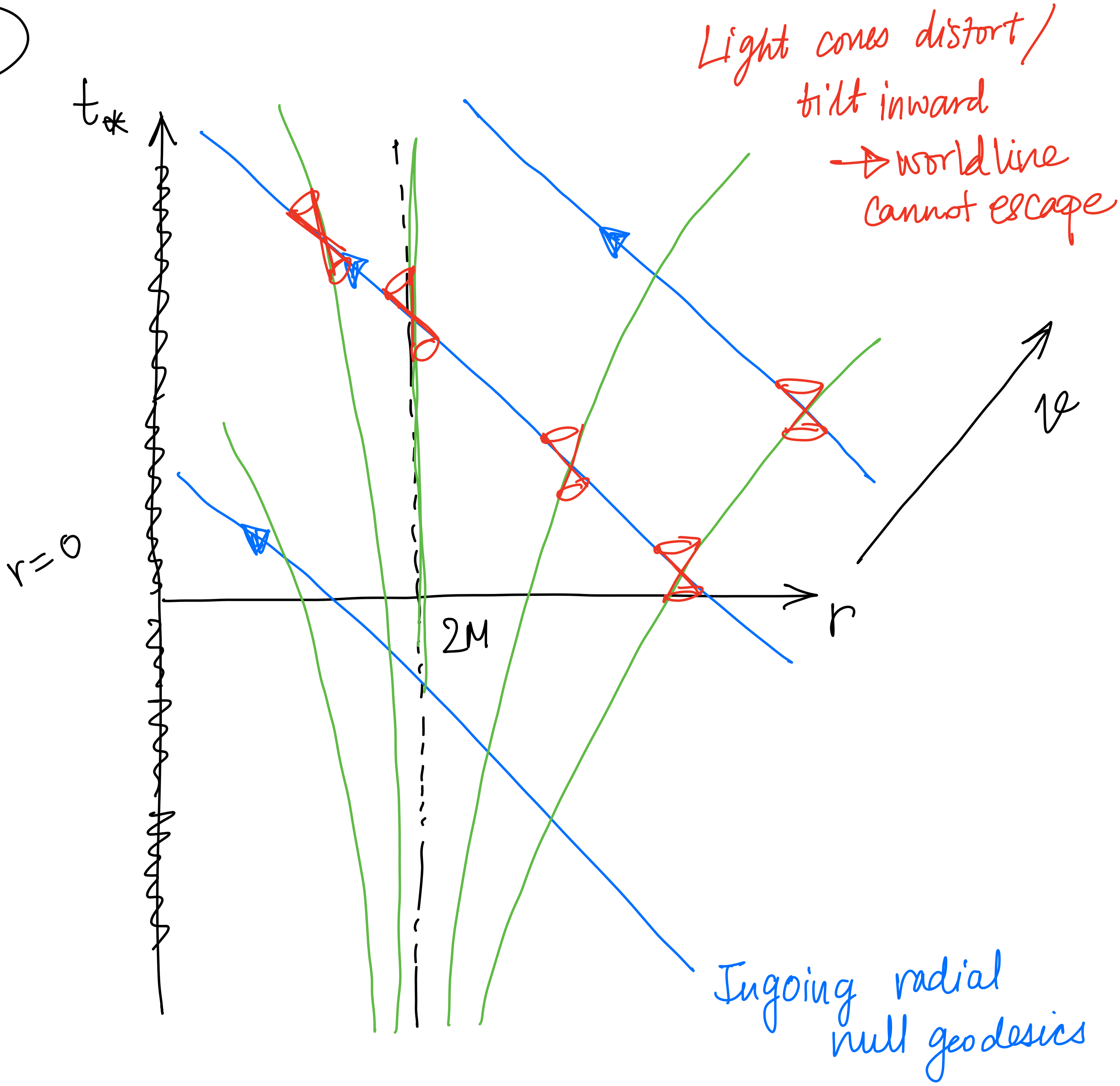


$t_* = v - r$

Outgoing null geodesics : $du = 0$, $u = t - r_* = \text{const.}$

$\Rightarrow$ in terms of (v, r, θ, φ): $\quad v = t + r_*$

$$= u + 2r_*$$

$$= 2r_* + \text{const.}$$

Another way to see that for $r < 2M$, worldlines cannot escape is :

Consider $r \leq 2M$, for causal geodesics (i.e. $ds^2 \leq 0$)

Then :
$$2\,dv\,dr = ds^2 + \left(1 - \frac{2M}{r}\right) dv^2 - r^2 d\Omega^2$$
$$= -\left[-ds^2 + \left(\frac{2M}{r} - 1\right) dv^2 + r^2 d\Omega^2\right]$$
$$\leq 0$$

Now : For future-directed worldline : $dv > 0$

$$\Rightarrow dr \leq 0$$

i.e. causal geodesics in $r < 2M$ can only travel in decreasing r direction.

r-direction is timelike in the interior

— Black hole

KVF : $k = \partial_t \quad (r > 2M)$

Now: $k = \frac{\partial x^\mu}{\partial t} \partial_\mu = \frac{\partial}{\partial v}$

Notice $k^2 = g_{\mu\nu} k^\mu k^\nu$
$= g_{vv} = \frac{2M}{r} - 1$

$\Rightarrow$ $r > 2M : k^2 < 0$ timelike
$r = 2M : k^2 = 0$ null
$r > 2M : k^2 > 0$ spacelike

Similarly, Outgoing EF coords: (u, r, θ, φ)

describes a White Hole $(r < 2M)$

"time reversed BH"

Can see: Substitute $u = -v$ in ds^2_{OutEF}

$\rightarrow$ isometric, up to sign of time orientation

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 - 2\,du\,dr + r^2 d\Omega^2$$

No causal curve can enter $r=2M$ from $r>2M$, i.e. $dr \geq 0$ for future-directed worldlines in $r<2M$.

$\rightarrow$ r can only increase.

KVF $k = \frac{\partial}{\partial u}$ in (u, r, θ, φ)

- Kruskal extension (U, V, θ, φ)

(Maximal analytic extension of Schw.)

Introduce new null coords.

$$U = -e^{-u/4M}, \quad V = e^{v/4M}$$

$(\partial/\partial U,\ \partial/\partial V$ null VFs$)$

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M}\, dU\, dV + r^2 d\Omega^2$$

$r(U, V)$ implicitly defined by:

$$UV = -e^{r_*/2M} = -e^{r/2M}\left(\frac{r}{2M} - 1\right)$$

Notice: $r = 2M \Rightarrow UV = 0 \Rightarrow U = 0$ or $V = 0$

$\longrightarrow$ two surfaces that intersect @ $U = 0,\ V = 0$.

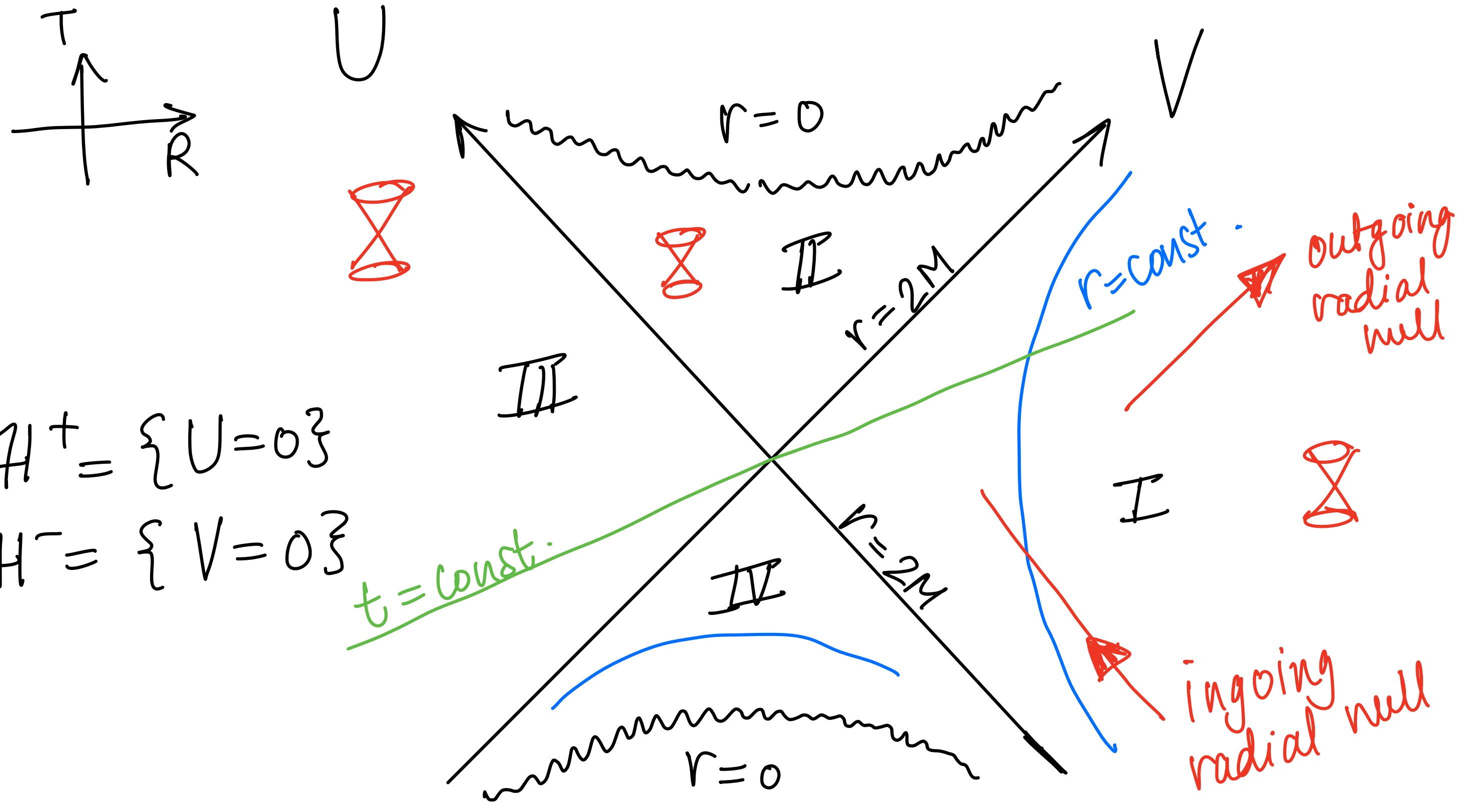
$r = 0 \Rightarrow UV = 1 \rightarrow$ hyperbola with two branches

For convenience, can also introduce timelike, spacelike coords:

$$T = \frac{1}{2}(V + U)\ , \quad R = \frac{1}{2}(V - U)$$

$$ds^2 = \frac{32M^3}{r} e^{-r/2M}\left(-dT^2 + dR^2\right) + r^2 d\Omega^2$$

where $r(T, R)$: $\quad T^2 - R^2 = \left(1 - \frac{r}{2M}\right) e^{r/2M}$



$\mathcal{H}^+ = \{U = 0\}$

$\mathcal{H}^- = \{V = 0\}$

- Light cones as in Mink. → very useful
  ∵ radial ingoing and outgoing are @ 45°
- Event horizons : $r = 2M$, $T = \pm R$
- $r = \text{const.} \iff T^2 - R^2 = \text{const.}$

I, II : Ingoing EF

I, IV : Outgoing EF

I, III : Isometric, exterior Schw. under $U, V \to -U, -V$

II : BH

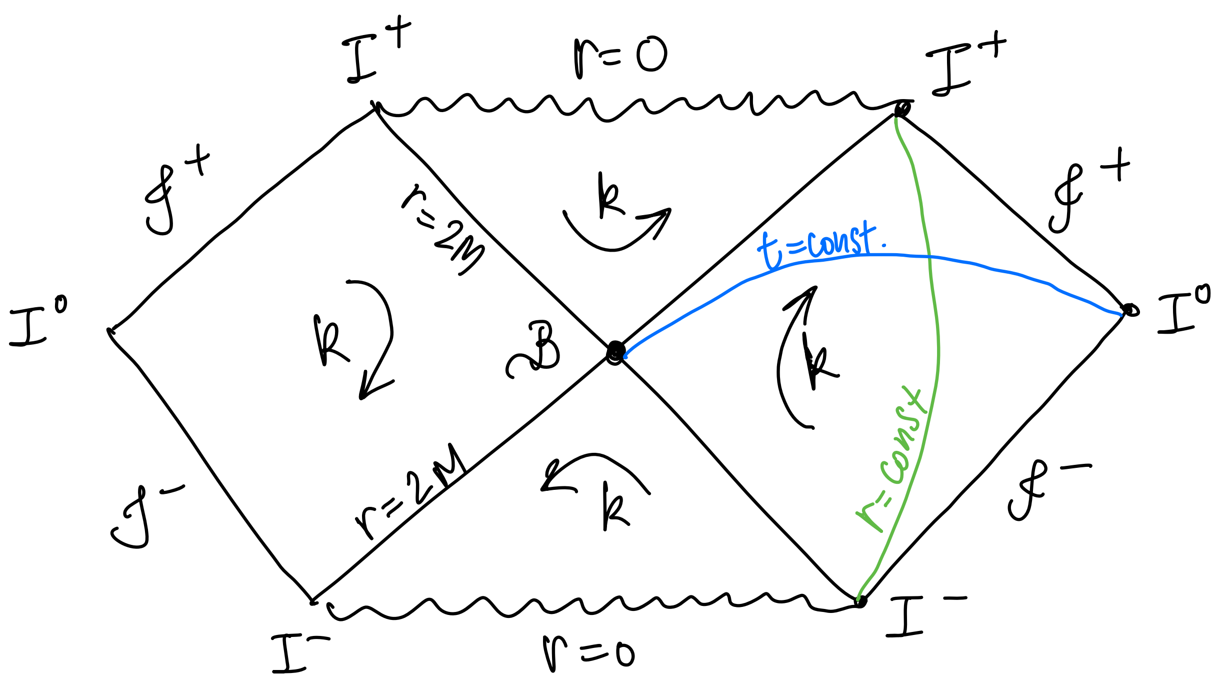
IV : WH

KVF: $k = \frac{1}{4M}\left(V\partial_V - U\partial_U\right)$

$k^2 = \frac{2M}{r} - 1$ (as before, as should be $\because$ scalar)

<u>Penrose diagram</u> : Bring ∞ to finite distance

Conformally compactify Kruskal manifold



$\mathcal{B}$: Bifurcation surface, here S^2

$$U=0, \; V=0, \; k=0$$

Killing Horizon (KH)

A null hypersurface $\mathcal{N}$ is a KH if $\exists$ a KVF ξ s.t., on $\mathcal{N}$, ξ is normal to $\mathcal{N}$.

Such ξ satisfies:

$$\nabla_\xi \xi \big|_{\mathcal{N}} = \underset{\text{surface gravity}}{\kappa} \xi$$

$$\kappa = \xi^\mu \partial_\mu \ln|f|$$

where $\xi = \underset{\text{arbitrary fn.}}{f} n$, $\nabla_n \underset{\text{normal to } \mathcal{N}}{n} = 0$

Also: $\nabla_\xi \kappa^2 = 0$ constant on orbits of ξ

See later: κ related to Hawking temperature $(\hbar\kappa/2\pi)$

Classical interpretation: κ is the acceleration of a static particle near $\mathcal{N}$ as measured @ spatial infinity.

E.g. for Kruskal:

Killing horizon $\mathcal{N} = \mathcal{H}^+ \cup \mathcal{H}^-$

with KVF $\xi = k = \frac{1}{4M}\left(V\partial_V - U\partial_U\right)$

$\mathcal{N}$: Bifurcate KH, $\mathcal{B} = \mathcal{H}^+ \cap \mathcal{H}^-$

On $\mathcal{H}^+$ (ie. $\{U=0\}$)

$$k = \frac{1}{4M} V\partial_V \quad , \quad \kappa = \frac{1}{4M}$$

On $\mathcal{H}^-$ (ie $\{V=0\}$)

$$k = \frac{-1}{4M} U\partial_U \quad , \quad \kappa = \frac{-1}{4M}$$

Kerr - Newman

Boyer - Lindquist Coords (t, r, θ, φ)

$$ds^2 = -\frac{(\Delta - a^2\sin^2\theta)}{\rho} dt^2 - 2a\sin^2\theta \frac{(r^2+a^2-\Delta)}{\rho} dt\, d\varphi$$

$$+ \left[\frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\rho}\right]\sin^2\theta\, d\varphi^2 + \frac{\rho}{\Delta} dr^2 + \rho\, d\theta^2$$

where: $\rho = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2Mr + a^2 + E^2$

Stationary, axisymm.

KVFs : $k = \partial_t$, $m = \partial_\varphi$

$$\begin{cases} \text{RN}: & a = 0 \\ \text{Kerr}: & \mathbb{E} = 0 \end{cases}$$

$a = \mathbb{J}/M$

angular momn.

Both have two types of horizons : $r_\pm$

$$\Delta = (r - r_+)(r - r_-)$$

RN $(M^2 > \mathbb{E}^2)$	Kerr $(M^2 > a^2)$
$r_\pm = M \pm \sqrt{M^2 - \mathbb{E}^2}$	$r_\pm = M \pm \sqrt{M^2 - a^2}$

r_+ : event horizon

r_- : Cauchy horizon $\longrightarrow$ $\bar{M}$ not globally hyperbolic

Surface gravities

$$\kappa_{\pm} = \frac{r_{\pm} - r_{\mp}}{2 r_{\pm}^2}$$

$$\kappa_{\pm} = \frac{r_{\pm} - r_{\mp}}{2(r_{\pm}^2 + a^2)}$$

Killing horizons $\mathcal{N}_{\pm} : r = r_{\pm}$

KVF

$\xi = k$

KVFs

$$\xi_{\pm} = k + \left(\frac{a}{r_{\pm}^2 + a^2}\right) m$$

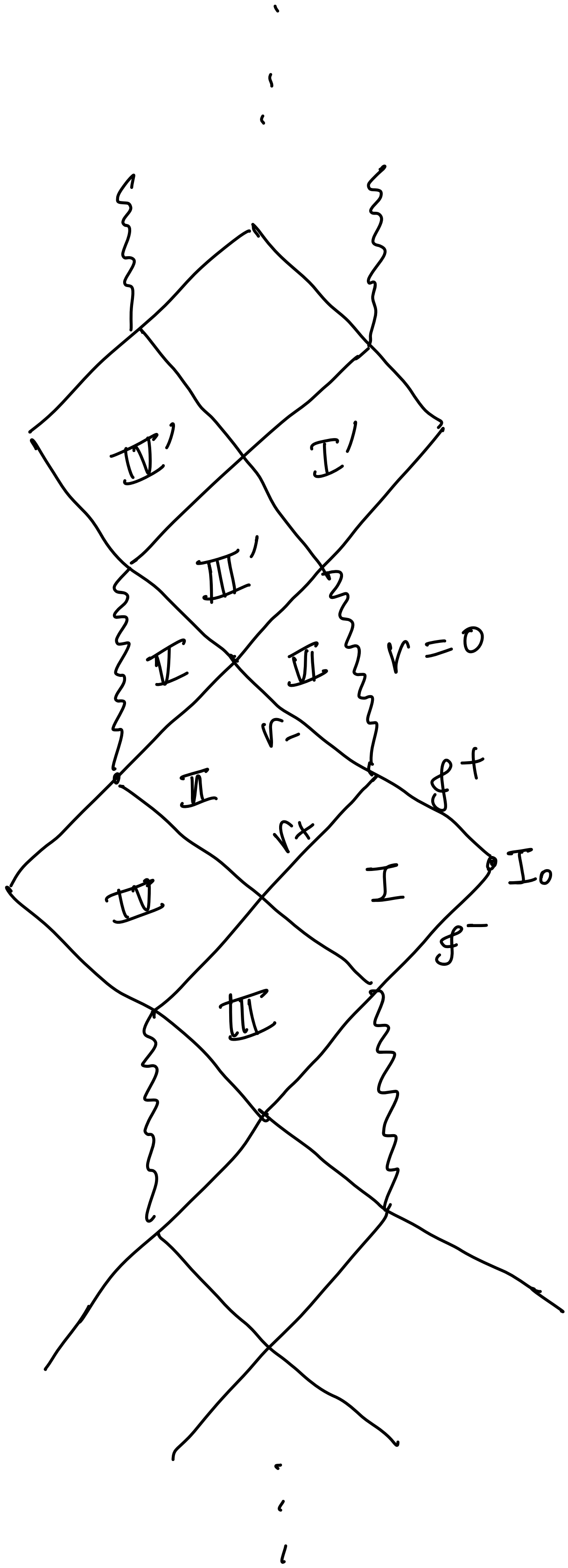
Event horizon r_+

$$\xi = k + \left(\underbrace{\frac{a}{r_+^2 + a^2}}\right) m$$

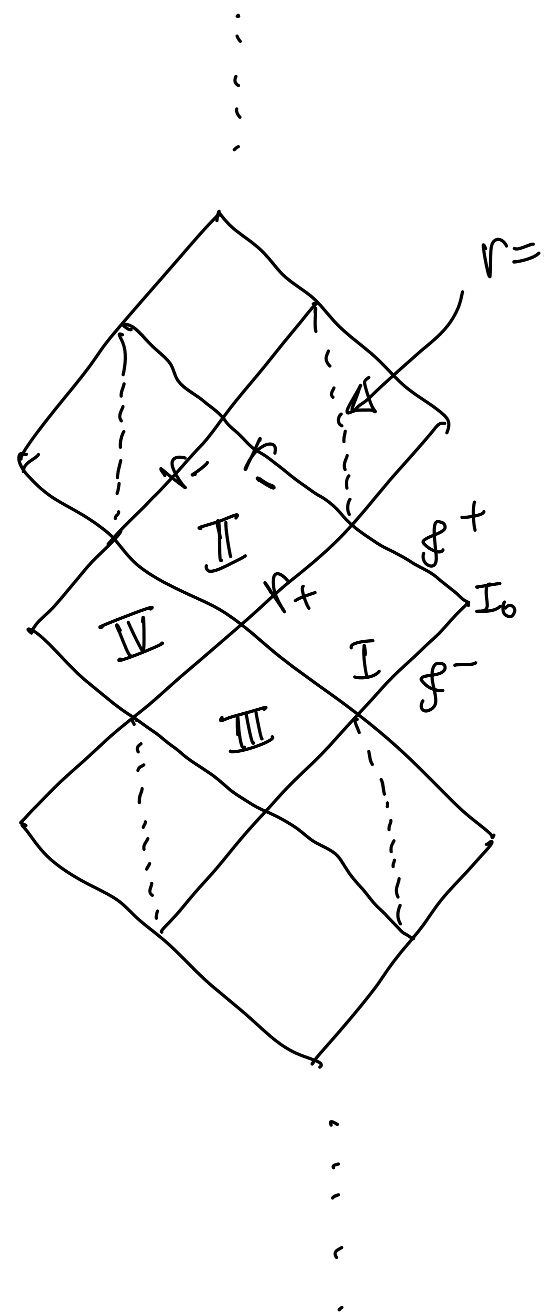
angular velocity of horizon Ω_H

RN

$(M^2 > E^2)$



Kerr
$(M^2 > a^2)$



Note : $M^2 = E^2$ " Extremal RN "

$M^2 = a^2$ " Extremal Kerr "

Further Reading

- Frolov, Novikov, "Black Hole Physics: Basic Concepts and New Developments", Springer book, DOI: 10.1007/978-94-011-5139-9.
- Townsend, "Black holes", Part III Lecture notes, [gr-qc/9707012]
- Reall, "Black holes", Part III Lecture notes
- Wald, "General Relativity", text book
- Hawking, Ellis, "The large-scale structure of space-time", text book