Diffeomorphism charges and GR

Lecture #8

In the last lecture, we looked at Noether's (first) theorem and the corresponding conserved charges associated with continuous Lagrangian symmetries — importantly, in the presence of boundaries. Briefly recall:

By a Lagrangian symmetry, we mean: a transformation generated by a vector field X s.t. L is invariant upto a boundary term, i.e.

$$\mathcal{L}_X L = d\ell_X \qquad (1)$$

where ℓ_X satisfies

$$\ell_X + I_X \delta\ell \Big|_{\mathcal{I}} = d\alpha_X \qquad (2)$$

for some $(D-2,0)$-form.

In other words,

E.o.M are invariant under the transformation. ("Symmetry")

Notice, here $X \in \mathfrak{X}(\mathcal{C})$ is a vector field on the configuration space, ie. space of dynamical fields. In other words, $\mathcal{L}_X$ acts on dynamical fields only in L above.

In particular, boundary condition (2) ensures that, on-shell, the action is invariant up to terms on $\Sigma_\pm$

i.e. $$\mathcal{L}_X S = \int_{\Sigma_+ - \Sigma_-} l_X + I_X \delta l - d\alpha_X \quad (3)$$

Conserved charges (generate non-trivial transfs. on solution space)

$$H_X = \int_{\Sigma} I_X \theta - l_X + \int_{\partial\Sigma} \alpha_X - I_X C \qquad (4)$$

where C is a $(D-2)$-form that we

introduced a couple lectures back via the following b.c.

$$\theta + \delta \ell \Big|_{\mathcal{I}} = dC \qquad (5)$$

ensuring that δS is invariant up to terms on $\Sigma_\pm$ on-shell, i.e.

$$\delta S = \int_{\Sigma_+ - \Sigma_-} \theta + \delta \ell - dC \qquad (6)$$

where here the generator of the variation is not specified in general. In this lecture, our focus will be variations generated by spacetime diffeomorphisms, i.e. vector fields $\xi \in \mathfrak{X}(M)$ on spacetime; and later, to detail the specific, important, example of GR.

Covariant tensor fields

Consider a tensor field T on field space $\mathcal{C}$.
Notice that T is also a tensor field on spacetime M, locally constructed out of dynamical and background fields in general; so, on spacetime we can write it as $T(\phi;\gamma)$ in general. Let $\xi \in \mathfrak{X}(M)$ be a vector field generating some diffeomorphism $F_\xi \in \mathrm{Diff}(M)$.

We are interested in variations $\delta_\xi T$ due to spacetime diffeomorphisms.

Recall that, by definition:

(could be dynamical or background)

- Given a tensor field β on M, then its variation under an infinitesimal diffeo. is

$$\delta_\xi \beta = \mathcal{L}_\xi \beta$$

Lie derivative on M

- Given a dynamical tensor field ϕ in $\mathcal{C}$, then its variation under a diffeo is

$$\delta_\xi \phi = \mathcal{L}_{X_\xi} \phi \qquad (7)$$

field space Lie derivative

where $X_\xi \in \mathfrak{X}(\mathcal{C})$ is the vector field lifted up from a diffeo-generating vector field $\xi^\mu \in \mathfrak{X}(M)$ on spacetime, given by

$$X_\xi = \int_M \mathcal{L}_\xi \phi \cdot \frac{\delta}{\delta \phi} \, . \qquad (8)$$

→ Notice that not every variation $\delta_\xi \beta$ can be written in terms of $\mathcal{L}_{X_\xi}$.

Then, we say that T is covariant (or, transforms covariantly) under a diffeomorphism F_ξ, if

$$\delta_\xi T = \mathcal{L}_\xi T \qquad (9)$$

i.e. if its variations on field space and spacetime match/coincide.

The simplest way for $T(\phi;\gamma)$ to be covariant under a given ξ (in the above sense) is for all background fields γ in its expression to be ξ-invariant, i.e. $\mathcal{L}_\xi \gamma = 0$. This ensures that ξ acts non-trivially only on the dynamical fields ϕ parts of T, so that this action can indeed be lifted up consistently to the field space, and thus get matched with $\mathcal{L}_{X_\xi}$

For example: Maxwell theory on fixed Minkowski background, as considered previously, with Lagrangian $L = -\frac{1}{2} F \wedge * F$. We know that L is a D-form, which means that it transforms as a D-form under diffeomorphisms acting on both the dynamical and background fields. But the point here is that L is covariant under Poincaré transformations, because these are the isometries of the Minkowski metric (thus keeping the background structure $\eta_{\mu\nu}$ invariant).

In fact, a similar statement will hold for any Lagrangian L, on a fixed background metric (and no other background fields), whence L would transform covariantly under the isometries of the background metric.

More generally, one could have ξ-invariant combinations of $\{\gamma\}$ in T, instead of every γ being individually ξ-invariant.

In fact, there could be theories with L for which it is not possible to find even a single diffeomorphism that leaves at least one of its fields (out of which L is constructed) invariant; in other words, L which only depends on fields that transform non-trivially under arbitrary (or all) diffeomorphisms. The only way in which such an L can transform covariantly is if all the fields on which it depends are dynamical, i.e. there are no background fields. This is background independence.

Prime example: Einstein-Hilbert Lagrangian in GR

We will return to GR later below.

Aside:

With eqn. (9) in place, it is clear how the variations δS (eqn. 6) and $\mathcal{L}_X S$ (eqn. 3), which we encountered during the past couple of lectures, are related:

For $S = \int_M L + \int_{\partial M} \ell$, and covariant L and ℓ under some infinitesimal diffeo ξ,

then $\quad \delta_\xi S = \mathcal{L}_{X_\xi} S$

i.e. S is covariant under ξ.

End of Aside

In addition to covariance of L and invariance of action (up to terms on $\Sigma_\pm$), transformation X_ξ must also be compatible with given b.c for it to be a symmetry.

Let's consider variation of $S = \int_M L + \int_{\partial M} \ell$

due to a ξ, under which L is given to be covariant:

$$\delta_\xi S = \underbrace{\int_M \delta_\xi L}_{} + \int_{\partial M} \delta_\xi \ell$$

$$= \int_M \mathcal{L}_\xi L = \int_M d\,\iota_\xi L$$

$$= \int_{\partial M} \iota_\xi L$$

Thus:

$$\delta_\xi S = \int_{\mathcal{I}} \iota_\xi L + \delta_\xi \ell + \int_{\Sigma_+ - \Sigma_-} \iota_\xi L + \delta_\xi \ell$$

But, only terms on $\Sigma_\pm$ are allowed in δS.

(recall: already have b.c. in eqns. (2) & (5))

Need additional boundary conditions to ensure that the contribution $\int_{\mathcal{I}}$ in the above eqn. vanishes (up to terms on $\Sigma_\pm$):

- Normal component of ξ^μ vanishes at $\mathcal{I}$, so that ξ does not move $\mathcal{I}$

  $$\Rightarrow \int_{\mathcal{I}} \iota_\xi L = 0.$$

- Covariance of ℓ under ξ on $\mathcal{I}$

  $$\Rightarrow \delta_\xi \ell\big|_{\mathcal{I}} = \mathcal{L}_\xi \ell\big|_{\mathcal{I}} = d\iota_\xi \ell\big|_{\mathcal{I}}$$

  $$\Rightarrow \int_{\mathcal{I}} \delta_\xi \ell = \int_{\mathcal{I}} d\iota_\xi \ell = \int_{\partial\mathcal{I}} \iota_\xi \ell$$

  allowed contribution on $\partial\Sigma_\pm$.

In fact, ℓ (and C) are required to be covariant under foliation-preserving diffeos

Diffeomorphism boundary charges

Our focus here has been on diffeomorphism symmetry, instead of a standard gauge symmetry, e.g. $A \to A + d\lambda$, like we saw in previous class. In this case then, there are additional subtleties that arise (as discussed above) due to the fact that the transformations under consideration here do not live only over field space, but are actually lifted (covariantly) from spacetime / the base space. As explained above, we must thus consider covariant tensors, like Lagrangian. We can adopt what we saw in the last lecture, to the present setting, as follows.

Essentially, now have:

$$\ell_X = \iota_\xi L \qquad (10)$$

$$\alpha_X = \iota_\xi \ell \qquad (11)$$

i.e. H_{X_ξ} independent of Σ.

Then, Conserved charge is now:

$$H_{X_\xi} = \int_\Sigma I_{X_\xi}\theta - \iota_\xi L + \int_{\partial\Sigma} \iota_\xi \ell - I_{X_\xi} C \qquad (12)$$

by substituting (10)-(11) into (4).

H_{X_ξ} is the generator of evolution in phase space corresponding to the diffeo F_ξ. This is the Noether charge associated with the diffeo F_ξ.

Noether current is:

$$J_\xi = I_{X_\xi}\theta - \iota_\xi L. \qquad (13)$$

Now, in the case when L is covariant under arbitrary diffeos (e.g. GR), and J_ξ is linear in ξ, then $\exists$ $(D-2)$-form Q_ξ s.t., on-shell:

$$J_\xi = dQ_\xi \,. \tag{14}$$

Then,

$$H_{X_\xi} = \int_{\partial\Sigma} Q_\xi + \iota_\xi \ell - I_{X_\xi} C$$

i.e. for generally covariant theories, the (Hamiltonian) generator of <u>any</u> continuous diffeo. is a <u>pure boundary</u> term (on-shell).

"surface charge"

Finally, on to GR.

General Relativity

Have Lagrangians

$$L = \frac{1}{16\pi G}(R - 2\Lambda)\,\varepsilon$$

$$\ell = \frac{1}{8\pi G} K\,\varepsilon_{\partial M}$$

where ε : volume form on M

$\varepsilon_{\partial M}$: " " " ∂M

R : spacetime Ricci scalar

Λ : cosmological constant

K : extrinsic curvature scalar $g_{\mu\nu}K^{\mu\nu}$

Metric $g_{\mu\nu}$ is dynamical, and there are no non-trivial background fields in L and ℓ.

(examples of trivial background fields: coupling constants, volume form)

$\rightarrow$ generally covariant, background independent theory

Vary L and ℓ:

$$\delta L = E^{\mu\nu}\delta g_{\mu\nu} + d\theta$$

where: $\underline{\underline{e.o.m}}$

$$E^{\mu\nu} = \frac{1}{16\pi G}\left(-R^{\mu\nu} + \frac{1}{2}R g^{\mu\nu} - \Lambda g^{\mu\nu}\right)\varepsilon$$

$$\theta = \iota_V \varepsilon \;, \quad V^\mu = \frac{1}{16\pi G}\left(g^{\mu\rho}\nabla^\sigma \delta g_{\rho\sigma} - g^{\rho\sigma}\nabla^\mu g_{\rho\sigma}\right)$$

and we have made use of:

$$\delta R = -R^{\mu\nu}\delta g_{\mu\nu} + \nabla^\mu\nabla^\nu \delta g_{\mu\nu} - \nabla_\rho\nabla^\rho g^{\mu\nu}\delta g_{\mu\nu}$$

$$\delta\varepsilon = \left(\frac{1}{2}g^{\mu\nu}\delta g_{\mu\nu}\right)\varepsilon$$

Also,

$$\delta\ell = \frac{1}{16\pi G}\Big[\left(K h^{\mu\nu} - K^{\mu\nu}\right)\delta g_{\mu\nu} + g^{\rho\sigma} n^\mu \nabla_\mu g_{\rho\sigma} - n^\mu \nabla^\nu \delta g_{\mu\nu} - D_\mu\left(h^{\mu\nu} n^\rho \delta g_{\nu\rho}\right)\Big]\varepsilon_{\partial M}$$

where we have used the following variations:

$$\delta \varepsilon_{\partial M} = \left(\frac{1}{2} h^{\mu\nu} \delta g_{\mu\nu}\right) \varepsilon_{\partial M}$$

$$\delta n_\mu = \frac{1}{2} n^\rho \left(\delta^\sigma_\mu - h^\sigma_\mu\right) \delta g_{\rho\sigma}$$

∂M induced metric

$$\delta K = -\frac{1}{2} K^{\mu\nu} \delta g_{\mu\nu} + \frac{1}{2} g^{\mu\nu} n^\rho \nabla_\rho \delta g_{\mu\nu}$$

$$- \frac{1}{2} n^\mu \nabla^\nu \delta g_{\mu\nu} - \frac{1}{2} D_\mu \left(h^{\mu\nu} n^\rho \delta g_{\nu\rho}\right)$$

∂M normal form

Covariant derivative on ∂M compatible with $h_{\mu\nu}$

Then, to find C, we need the pullback of θ to ∂M, ie.

$$\theta\big|_{\partial M} = \iota_V \varepsilon \big|_{\partial M} = \iota_V (n_\wedge \varepsilon_{\partial M})\big|_{\partial M}$$

$$= n_\mu V^\mu \varepsilon_{\partial M}$$

Then get:

$$\theta + \delta l \big|_{\partial M} = \frac{-1}{16\pi G} \left(K^{\mu\nu} - K h^{\mu\nu}\right) \varepsilon_{\partial M} \delta g_{\mu\nu} + dC$$

where $C = \imath_W \varepsilon_{\partial M}$,

$$W^\mu = \frac{-1}{16\pi G} h^{\mu\nu} n^\rho \delta g_{\nu\rho} .$$

To satisfy b.c. (5), thus need:

$$\left(K^{\mu\nu} - K h^{\mu\nu} \right) \delta g_{\mu\nu} \Big|_{\mathcal{I}} = 0$$

Again,
Dirichlet b.c: $h_\mu{}^\rho h_\nu{}^\sigma \delta g_{\rho\sigma} \big|_{\mathcal{I}} = 0$. (*)

Diffeos ξ satisfying (*) above are s.t.

- $\xi^\mu n_\mu \big|_{\mathcal{I}} = 0$
- $D_{(\mu} \xi_{\nu)} \big|_{\mathcal{I}} = 0$ (Killing eqn.)

  i.e. ξ approaches a Killing vector field of the boundary metric.

Such ξ are indeed foliation-preserving.

Noether current:

$$J_\xi = \iota_{j_\xi} \varepsilon$$

where $$j^\mu_\xi = \frac{1}{8\pi G}\Big[\nabla_\nu \nabla^{[\nu}\xi^{\mu]} + \left(R^{\mu\nu} - \frac{1}{2}R g^{\mu\nu} + \Lambda g^{\mu\nu}\right)\xi_\nu\Big].$$

Charge aspect:

$$Q_\xi = \frac{-1}{16\pi G} \star d\xi$$

For H_ξ, we need pull-back

$$Q_\xi\Big|_{\partial\Sigma} = \frac{-1}{16\pi G}\left(\tau^\mu n^\nu - \tau^\nu n^\mu\right)\nabla_\mu \xi_\nu \, \varepsilon_{\partial\Sigma}$$

where $\varepsilon = \tau \wedge n \wedge \varepsilon_{\partial\Sigma}$

τ: normal form on $\partial\Sigma$ (viewed as the boundary of its past in $\mathcal{I}$)

Also, have

$$\iota_\xi \ell \Big|_{\partial\Sigma} = \frac{-1}{8\pi G} \xi^\mu \tau_\mu K \, \varepsilon_{\partial\Sigma}$$

$$I_{X_\xi} C \Big|_{\partial\Sigma} = \frac{1}{16\pi G} \left(\tau^\mu n^\nu + \tau^\nu n^\mu \right) \nabla_\mu \xi_\nu \, \varepsilon_{\partial\Sigma}$$

Thus, get:

$$H_{X_\xi} = \frac{-1}{8\pi G} \int_{\partial\Sigma} \tau^\mu \xi^\nu \left(-K_{\mu\nu} + h_{\mu\nu} K \right) \varepsilon_{\partial\Sigma}$$

(Recall: $K_{\mu\nu} = h_\mu{}^\rho \nabla_\rho n_\nu$)

$\longrightarrow$ Have recovered correct generator of boundary isometry with Killing vector field ξ^μ.

Next: Black holes

Further Reading

- Compère, Fiorucci, "Advanced lectures on GR" [arXiv: 1801.07064]
- Harlow, Wu, "Covariant phase space with boundaries" [arXiv: 1906.08616]
- Lee, Wald, 1990, "Local symmetries and Constraints"
- Wald, "Black hole entropy is the Noether charge" [gr-qc/9307038]
- Iyer, Wald, 1994, "Some properties of the Noether charge and a proposal for dynamical black hole entropy"