6 Gauge symmetry and constraints in field theory

Let us see how the covariant phase space construction goes for two examples. In both cases we will consider fields in flat (i.e. Minkowski) spacetime.

Scalar field

As we have already seen, for a scalar field of mass m the presymplectic potential density is

$$\theta = \delta\phi * \mathrm{d}\phi\,, \tag{6.1}$$

In Minkowski coordinates t, x^i, we have Cauchy surfaces Σ at fixed values of t. Note that $\partial\Sigma$ is at spacelike infinity $r = \sqrt{\delta_{ij}x^i x^j} \to \infty$. In order to regularise the construction, let us put $\partial\Sigma$ at finite radius $r = R$. Then at the end we can take $R \to \infty$. Let us also take $l = 0$.

We thus have

$$\int_{[t_0,t_1]\times\partial\Sigma} \theta = \int \mathrm{d}t \int^{S_{D-2}} \mathrm{d}^{D-2}\psi\, \sqrt{\gamma} R^{D-2}\, \delta\phi\, \partial_r\phi. \tag{6.2}$$

Here, ψ are coordinates on the sphere S^{D-2} at $r = R$, and γ is the unit metric on the $(D-2)$-sphere. Note that if we restrict to ϕ such that

$$\phi = \mathcal{O}\Big(\frac{1}{r^{(D-2)/2}}\Big) \tag{6.3}$$

for large r, then we have

$$\int_{[t_0,t_1]\times\partial\Sigma} \theta = \mathcal{O}\Big(\frac{1}{R}\Big) \to 0 \quad \text{as} \quad R \to \infty. \tag{6.4}$$

So in this case we can write $\theta + \delta l \to 0$ on $[t_0, t_1] \times \partial\Sigma$, and the variational principle will be well-defined. This kind of boundary condition, restricting the behaviours of fields in certain regions at infinity, is known as an asymptotic fall-off condition.

We then have

$$\Theta_\Sigma = \int_\Sigma \theta = \int_0^R \mathrm{d}r\, r^{D-2} \int_{S^{D-2}} \mathrm{d}^{D-2}\psi\, \delta\phi\, \partial_t\phi. \tag{6.5}$$

Then we have the presymplectic form

$$\Omega_\Sigma = \int_\Sigma \delta\theta = \int_0^R \mathrm{d}r\, r^{D-2} \int_{S^{D-2}} \mathrm{d}^{D-2}\psi\, \delta\phi\, \partial_t\delta\phi. \tag{6.6}$$

Let us be explicit about the components of this configuration space 2-form. If $\delta_{1,2}\phi$ are the components of two configuration space vectors, then we have

$$\Omega_\Sigma(\delta_1\phi, \delta_2\phi) = \int_0^R \mathrm{d}r\, r^{D-2} \int_{S^{D-2}} \mathrm{d}^{D-2}\psi\, (\delta_1\phi\, \partial_t\delta_2\phi - \delta_2\phi\, \partial_t\delta_1\phi). \tag{6.7}$$

Actually, on the space of solutions obeying (6.3), Ω_Σ is a symplectic form, because it is non-degenerate. To see this, note that the equation of motion for ϕ is just the wave equation $(\Box - m^2)\phi = 0$, and it is well-known that the solutions of the wave equation are uniquely determined by the values of ϕ and $\partial_t\phi$ on Σ. Thus for any $\delta_1\phi$ we can always find a $\delta_2\phi$ such that $\Omega_\Sigma(\delta_1\phi, \delta_2\phi)$ is non-zero.

Maxwell field

Pure Maxwell theory consists of an electromagnetic gauge potential 1-form field $A = A_\mu \, dx^\mu$. The Lagrangian density may be written

$$L = \frac{1}{2} F \wedge *F, \tag{6.8}$$

where $F = dA$ is the field strength 2-form. Upon varying $A \to A + \delta A$, one finds

$$\delta L = d\delta A \wedge *F = \delta A \wedge d * F + d(\delta A \wedge *F)\,, \tag{6.9}$$

So the Maxwell equations of motion are $d * F = 0$[4], while the presymplectic potential density is $\theta = \delta A \wedge *F$.

Thus, under an appropriate choice of boundary conditions, the presymplectic potential is

$$\Theta_\Sigma = \int_\Sigma \delta A \wedge *F, \tag{6.10}$$

while the presymplectic 2-form is

$$\Omega_\Sigma = \int_\Sigma \delta A \wedge \delta(*F). \tag{6.11}$$

The components of Ω_Σ are then

$$\Omega_\Sigma(\delta_1 A, \delta_2 A) = \int_\Sigma \Big(\delta_1 A \wedge \delta_2(*F) - \delta_2 A \wedge \delta_1(*F)\Big). \tag{6.12}$$

In this case, the presymplectic 2-form is *not* a symplectic form on the space of solutions, because it is degenerate. Indeed, suppose A solves the equations of motion. Then $A + d\lambda$ also solves the equations of motion, for any arbitrary function λ, because F does not change – this is a Maxwell gauge transformation. If we pick $\delta_1 A = d\lambda$, then, using the fact that $0 = \delta_2(d * F) = d\delta_2(*F)$ on-shell, we have

$$\Omega_\Sigma(\delta_1 A, \delta_2 A) = \int_\Sigma d\lambda \wedge \delta_2(*F) = \int_\Sigma d(\lambda\, \delta_2(*F)) = \int_{\partial\Sigma} \lambda\, \delta_2(*F). \tag{6.13}$$

Since λ is arbitrary, we can pick for it to vanish on $\partial\Sigma$[5] (but remain non-zero away from $\partial\Sigma$). Then $\Omega_\Sigma(\delta_1 A, \delta_2 A) = 0$ even though $\delta_1 A \neq 0$. Thus Ω_Σ is degenerate.

As a final point, note that the presymplectic 2-form can be written

$$\Omega_\Sigma = \int_\Sigma d^{D-1}x\, \delta A_i \delta E^i, \tag{6.14}$$

where

$$E^i = F^{it} = \partial_t A_i - \partial_i A_t \tag{6.15}$$

is a vector field on Σ usually called the electric field. Comparing this to the general version of the presymplectic 2-form

$$\Omega = \int_\Sigma d^{D-1}x\, \delta\varphi \cdot \delta\pi, \tag{6.16}$$

we can identify the initial data φ for the electromagnetic field as A_i, and the conjugate momentum π as E^i.

[4] The other part of Maxwell's equations, $dF = 0$, just come from $F = dA$ and $d^2 = 0$. In coordinates, these equations are equivalent to $\partial_\mu F^{\mu\nu} = 0$ and $\partial_{[\mu} F_{\nu\rho]} = 0$ respectively.

[5] In a later section, we will address what happens when λ doesn't vanish on $\partial\Sigma$.

6.1 Constraints and redundancy

In the canonical Hamiltonian approach to field theory, the state of a field is determined by initial data and conjugate momenta on an initial Cauchy surface Σ. We have seen how for a scalar field ϕ the initial data is the value taken at every point on Σ, while the conjugate momentum is the normal derivative $\partial_t\phi$ to Σ. We can freely specify each of these as two independent functions of Σ (so long as they obey the boundary conditions).

However, this is not always the case. Consider the Maxwell field, whose initial data and conjugate momenta are the spacelike components of the gauge potential A_i and the electric field E^i respectively. The gauge potential is freely specifiable, but the electric field must obey the so-called Gauss constraint

$$\nabla \cdot E = \partial_i E^i = 0. \tag{6.17}$$

Any electric field which does *not* obey the Gauss constraint is simply physically impossible, because the Gauss constraint is just one component of the equations of motion:

$$\partial_i E^i = \partial_\mu F^{\mu t} = 0. \tag{6.18}$$

So if the Gauss constraint is not obeyed, E^i cannot possibly be part of any solution to the equations of motion, and therefore cannot be part of a physically self-consistent classical system. In fact, this is a general pattern for the constraints in covariant field theories – they always arise as certain components of the equations of motion.

As we have explained previously, it is a deep property of physical systems that the existence of constraints can be associated with non-deterministic evolution of the state of the system. That is, given some initial data and conjugate momenta which obey the constraints, the subsequent evolution of those objects is not uniquely determined. Indeed, there will be an infinite family of different trajectories which are all consistent with the initial data and momenta.

The physical interpretation of this is that all these trajectories are physically equivalent to each other. Thus, there is a redundancy in the description of the system. Moreover, there will be differences in the final data and conjugate momenta at the end of the evolution of two separate, yet physically equivalent, trajectories. But since the initial state was the same, these data and momenta must describe the same physical state. So there is also a redundancy in the initial data and conjugate momenta as the specification of a physical state. The transition between two physically equivalent sets of initial data and conjugate momenta, or two physically equivalent trajectories, is known as a gauge transformation.

6.2 Gauge ‘symmetry’ and locality

This redundancy in the description of the physics is often called ‘gauge symmetry’, but this is a bit of a misnomer. Indeed, a symmetry of a system is some transformation that acts on the physical state, and can change that physical state. A gauge transformation, on the other hand, justs acts on whatever particular description of the physical state we happen to be using, and does *not* change the physical state. Symmetries are an intrinsic property of the physical system. Gauge symmetries are *not*; they just reflect the inefficiencies/redundancies of whatever method we are using to specify the physical state. If we have two different methods of specifying the state, both capable of fully characterising the physical properties of the system, those two different methods can have entirely different sets of gauge symmetries – but both methods are equally physically viable.

In fact, if we use the right kind of description of the state, we can eliminate the need for gauge symmetries entirely. We just need to make sure to describe everything in terms of physical observables – i.e. those that are uniquely determined for each physical state. The question then arises, why ever use a description that needs gauge symmetries? Wouldn't it be simpler to just use a non-redundant description to start with?

The main reason in field theory is that it is often impossible to use a description which is both local and non-redundant. For example, the local description of the electromagnetic field in terms of the gauge potential 1-form A has the gauge redundancy $A \to A + \mathrm{d}\lambda$ for functions λ. To eliminate this redundancy, we might hope to just describe the electromagnetic field in terms of the field strength $F = \mathrm{d}A$, which is still a local object. However, this is not enough, because there are certain physical observables which are not determined by F alone. In particular, consider the 'Wilson loop'

$$W[\gamma] = \int_\gamma A, \tag{6.19}$$

where γ is some closed spacetime curve. It is clear that $W[\gamma]$ is invariant under $A \to A + \mathrm{d}\lambda$, since

$$W[\gamma] = \int_\gamma A \to \int_\gamma (A + \mathrm{d}\lambda) = \int_\gamma A + \underbrace{\int_{\partial\gamma} \lambda}_{=0} = \int_\gamma A = W[\gamma]. \tag{6.20}$$

For contractible γ, there is a U such that $\gamma = \partial U$, and we have

$$W[\gamma] = \int_U F, \tag{6.21}$$

so given F we can compute this Wilson line. But for non-contractible γ, we cannot write $W[\gamma]$ in terms of F. Such Wilson lines are *non-local* observables that we would need to include as extra physical variables in our non-redundant description.

Non-local objects are somewhat more difficult to deal with than local objects in general. In fact, the Wilson loop is probably the least complicated kind of non-local object that arises in field theory, and using it directly already carries many subtleties and nuances. So there is a competition between the two types of description, both having their own advantages and disadvantages. The local description is easier to use, but involves gauge redundancies. On the other hand, the non-local description can be completely free of gauge redundancy, but is more difficult to set up and use directly.

Almost always when dealing with field theories we will use the local approach. Then, at the end, we only consider the physical observables, which are often non-local.[6] At some point in between these two stages, there has to be some restriction or refinement of the description that goes from the redundant local description to the non-redundant non-local description.

We already saw in the first couple of lectures how this works in the canonical approach. One imposes the primary constraints, then imposes stability conditions to get secondary constraints, and so on. In the rest of this section, we will describe how this works in the the covariant approach, and we will also describe how the two approaches are related.

[6] In gravity, something especially dramatic happens – essentially all physical observables are non-local. We will explore this further throughout the course.

6.3 The covariant kinematical phase space

Recall that in constructing the physical phase space in the canonical Hamiltonian approach, one first starts with the kinematical phase space, which is the space of all possible configurations of initial data and conjugate momenta, even those which do not obey the constraints. Let us now see how this space arises in the covariant approach.

In the previous section, we constructed the object

$$\Omega_\Sigma = \int_\Sigma \omega, \tag{6.22}$$

where Σ is a Cauchy surface, and ω is a $(D-1,2)$-form obeying

$$\omega = \delta\theta, \qquad \delta L = E \cdot \delta\phi + \mathrm{d}\theta\,. \tag{6.23}$$

Previously, we have been viewing Ω_Σ as a 2-form on the space of solutions $\mathcal{S}$. However, we may instead view it as a 2-form on configuration space $\mathcal{C} = \Gamma(\Phi)$, and we will do so now.

Ω_Σ on $\mathcal{C}$ is an exact form, so it is closed. However, it is not non-degenerate, so it is not a symplectic form. Even in the absence of gauge symmetries, Ω_Σ is only a presymplectic 2-form, since we have not yet imposed the equations of motion. As a consequence the fields away from Σ may be specified completely independently of those on Σ. Thus, any field variation $\delta\phi \in \mathfrak{X}(\mathcal{C})$ with support away from Σ will give $\Omega_\Sigma(\delta\phi) = 0$ (since ω is local).

We would like to construct a symplectic space from the presymplectic space $(\mathcal{C}, \Omega_\Sigma)$. There is a standard method for doing so, called 'gauge reduction'.[7] Let us describe this now.

Let (C, Ω) be a presymplectic manifold, and suppose that $X, Y \in \mathfrak{X}(C)$ are degenerate directions of Ω, i.e. vector fields on C satisfying

$$\Omega(X) := \iota_X\Omega = 0, \qquad \Omega(Y) := \iota_Y\Omega = 0. \tag{6.24}$$

Note that this implies

$$\mathcal{L}_X\Omega = \mathrm{d}(\underbrace{\iota_X\Omega}_{=0}) + \iota_X(\underbrace{\mathrm{d}\Omega}_{=0}) = 0, \tag{6.25}$$

and similarly $\mathcal{L}_Y\Omega = 0$. Then, using the general formula $[\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]}$, we have

$$0 = \mathcal{L}_X(\iota_Y\Omega) = \iota_Y(\underbrace{\mathcal{L}_X\Omega}_{=0}) + \iota_{[X,Y]}\Omega. \tag{6.26}$$

So $[X, Y]$ is also a degenerate direction of Ω. Thus the set of degenerate directions is closed under Lie brackets – such a set of vector fields is sometimes said to be in 'involution'. At this point we can use this property to invoke a theorem from differential topology known as Frobenius' theorem, which in this case implies that the set of degenerate directions consists of all vector fields tangent to a regular foliation of C. The set P of leaves of this foliation is the symplectic manifold[8] that we desire. To define the symplectic form $\overline{\Omega}$ on P, consider the map

[7] The procedure has this name for historical reasons, even though we are not reducing over gauge symmetries in the usual sense in this context.

[8] We will refer to P as a manifold, but actually this is somewhat non-rigourous. It is not true in general that P has the correct smooth structure for this to be a valid designation. Nevertheless, it has *most* of the correct structure, and this is good enough for now.

$\eta : C \to P$ that takes a point $x \in C$ to the leaf in which it resides. Then for any two vectors $V, W \in \mathrm{T}P$, we set

$$\overline{\Omega}(V, W) = \Omega(X, Y), \qquad \text{where } V = \eta_* X, W = \eta_* Y. \tag{6.27}$$

One can always find X, Y that satisfy the right hand side. Such X, Y are not unique, but the value of $\overline{\Omega}(V, W)$ is uniquely fixed by this definition, because X, Y can only change by degenerate directions of Ω.

Thus, from the presymplectic manifold (C, Ω) we obtain the 'gauge-reduced' symplectic 'manifold' $(P, \overline{\Omega})$. Let us apply this to the configuration space $(C, \Omega) = (\mathcal{C}, \Omega_\Sigma)$. As we noted above, the degenerate directions of Ω_Σ are simply those field variations $\delta\phi$ which do not have support on Σ (as detectable by $\omega|_\Sigma$). Thus, two field configurations ϕ_1, ϕ_2 are in the same leaf of the foliation of $\mathcal{C}$ if they agree on Σ. To be more precise, let us write Ω_Σ in the form

$$\Omega_\Sigma = \int_\Sigma \mathrm{d}^{D-1}x \, \delta\varphi \cdot \delta\pi. \tag{6.28}$$

φ, π are the initial data and conjugate momenta respectively, and are certain parts of the fields on Σ. Two configurations ϕ_1, ϕ_2 are in the same leaf if and only if they have the same φ, π. We can therefore use φ, π as a unique label for the leaf. The gauge-reduced symplectic form $\overline{\Omega}_\Sigma$ on the set of leaves $\mathcal{P}_{\text{kin.}}$ then takes exactly the same form as Ω_Σ in (6.28).

In going from $(\mathcal{C}, \Omega_\Sigma)$ to $(\mathcal{P}_{\text{kin.}}, \overline{\Omega}_\Sigma)$, we have eliminated all of the degrees of freedom away from Σ, and we are only left with a space $\mathcal{P}_{\text{kin.}}$ parametrising those degrees of freedom on Σ corresponding to the initial data and conjugate momenta. This space is exactly the same as the kinematical phase space of the canonical approach. Moreover, the symplectic form $\overline{\Omega}_\Sigma$ is exactly the same as that in the canonical approach.

6.4 Imposing the constraints

To get from the kinematical phase space to the physical one, in the canonical approach we had to impose the constraints. What does this look like from the covariant point of view? As we said at the beginning of this section, in a covariant field theory the constraints are just certain components of the equations of motion that restrict the initial data and conjugate momenta. Thus, to impose the constraints we just need to find some way to find a subspace of $\mathcal{P}_{\text{kin.}}$ that satisfies the equations of motion.

This is not so difficult if we use a geometric perspective. First, note that the space of solutions is naturally embedded in the configuration space by a map $i : \mathcal{S} \to \mathcal{C}$. Moreover, from the gauge-reduction to the kinematical phase space we have the map $\eta : \mathcal{C} \to \mathcal{P}_{\text{kin.}}$. Composing these two maps yields

$$\eta_{\mathcal{S}} = \eta \circ i : \mathcal{S} \to \mathcal{P}_{\text{kin.}}. \tag{6.29}$$

This map takes each solution to the equations of motion to its corresponding point in the kinematical phase space, i.e. its initial data and conjugate momenta on Σ. Let

$$\mathcal{P} = \mathrm{im}(\eta_{\mathcal{S}}) \tag{6.30}$$

be the image of this map. The space $\mathcal{P}$ consists of all possible initial data and conjugate momenta which can be part of a solution to the equations of motion, i.e. which are physically viable. In a theory with non-trivial constraints, $\mathcal{P}$ will be a proper subspace of $\mathcal{P}_{\text{kin.}}$.

We refer to $\mathcal{P} \subset \mathcal{P}_{\text{kin.}}$ as the constraint surface. Let $\underline{\Omega}_\Sigma$ be the pullback of $\overline{\Omega}_\Sigma$ to the constraint surface. Even though $\overline{\Omega}_\Sigma$ is non-degenerate by construction as a 2-form on $\mathcal{P}_{\text{kin.}}$, in general the 2-form $\underline{\Omega}_\Sigma$ on $\mathcal{P}$ will be degenerate. The degenerate directions of $\underline{\Omega}_\Sigma$ in this case correspond exactly to the traditional gauge symmetries on the canonical data φ, π.

To get the physical phase space, we need to carry out another round of gauge reduction as described in the previous subsection. The result is a space $\mathcal{P}_{\text{phys.}}$ of leaves of a foliation of $\mathcal{P}$, and a symplectic form $\Omega_{\text{phys.}}$ on $\mathcal{P}_{\text{phys.}}$.

6.5 Bypassing the splitting

Let us summarise the steps that went into going from configuration space to the physical phase space, using the constraints. First, we gauge reduced configuration space with the presymplectic form Ω_Σ. Then we imposed the constraints by restricting to solutions of the equations of motion using the map $\eta_S = \eta \circ i$. Finally we did gauge reduction a second time.

$$(\mathcal{C}, \Omega_\Sigma) \xrightarrow{\text{gauge reduction}} (\mathcal{P}_{\text{kin.}}, \overline{\Omega}_\Sigma) \xrightarrow{E=0} (\mathcal{P}, \underline{\Omega}_\Sigma) \xrightarrow{\text{gauge reduction}} (\mathcal{P}_{\text{phys.}}, \Omega_{\text{phys.}}). \tag{6.31}$$

This really should be thought of as a fairly covariant description of the canonical approach.

This procedure involved two rounds of gauge reduction. It also involved a splitting of the equations of motion: some components of the equations of motion became the constraints, while the remaining components must govern the dynamics (of the Cauchy slice). But in a covariant field theory, all of the components of the equations of motion ought to be treated on an equal footing. Indeed, in the covariant phase space formalism the final symplectic form shouldn't depend on the choice of Cauchy surface Σ – so there is no physical special role played by the components of the equations of motion that restrict the form of the fields at Σ.

Thankfully, there is a way to account for this within the covariant phase space formalism that both removes the arbitrariness of the splitting between the constraints and the dynamics, and makes the construction simpler over all. This is to first set the equations of motion to zero, i.e. to restrict to the space of solutions $\mathcal{S} \subset \mathcal{C}$, and only afterwards perform gauge reduction. It can be shown that the end result is exactly the same as the above procedure:

$$(\mathcal{C}, \Omega_\Sigma) \xrightarrow{E=0} (\mathcal{S}, \Omega_\Sigma) \xrightarrow{\text{gauge reduction}} (\mathcal{P}_{\text{phys.}}, \Omega_{\text{phys.}}). \tag{6.32}$$

In this way, we only need one round of gauge reduction. We 'bypass' the constraints. This is conceptually more useful, because there is a clear distinction between the equations of motion and the gauge reduction, and there is no arbitrary non-covariant splitting of the equations of motion.

On the other hand, there are situations where it can be practically more useful to employ (6.31) (for example, when quantising the theory). So it is useful to know about both, and indeed we will use both during the course.

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