4 Covariant field theories

Consider a classical particle. In a Hamiltonian treatment, at each moment in time the state of the particle can be described by finitely many numbers: the generalised coordinates q_a and conjugate momenta p^a, $a = 1, \dots, n$. The action for such a particle then takes the form

$$S = \int_0^T L(p, q, \dot{q})\,\mathrm{d}t\,, \quad \text{where} \quad L(p, q, \dot{q}) = \sum_a p^a \dot{q}_a - H(p, q), \tag{4.1}$$

and the equations of motion associated with this action are just Hamilton's equations

$$\dot{q}_a = \frac{\partial H}{\partial p^a}, \quad \dot{p}^a = -\frac{\partial H}{\partial q_a}. \tag{4.2}$$

The theory of the particle is 1-dimensional. At each moment in time, the particle lives at a 0-dimensional point somewhere along its worldline.

In this course, we want to consider field theories in an arbitrary number of spacetime dimensions D. It is possible to fit such theories into the framework we just described. At each moment in time, the fields live on a $(D-1)$-dimensional spacelike surface, which we will call Σ. The state of the fields at a given time are then specified by infinitely many numbers: the values of the fields φ_a at each point in Σ, as well as their conjugate momenta π^a. We upgrade the sum over a to include an integral over Σ:

$$\sum_a p^a \dot{q}_a \to \int_\Sigma \left(\sum_a \pi^a \dot{\varphi}_a \right) \mathrm{d}^{D-1}x\,. \tag{4.3}$$

The Hamiltonian is a functional of the fields and conjugate momenta, which we will assume can be written as a local integral:

$$H(p, q) \to H[\varphi, \pi] = \int_\Sigma \mathscr{H}(\varphi, \pi)\,\mathrm{d}^{D-1}x\,. \tag{4.4}$$

The action therefore takes the form

$$S = \int_0^T \mathrm{d}t \int_\Sigma \left(\sum_a \pi^a \dot{\varphi}_a - \mathscr{H}(\varphi, \pi) \right) \mathrm{d}^{D-1}x\,. \tag{4.5}$$

If we vary this action, we will find the field theory version of Hamilton's equations:

$$\dot{\varphi}_a = \frac{\delta \mathscr{H}}{\delta \pi^a}, \quad \dot{\pi}^a = -\frac{\delta \mathscr{H}}{\delta \varphi_a}. \tag{4.6}$$

This perspective on field theory, called the 'canonical' formulation, can sometimes be very useful. However, most of the time when doing field theory we do not deal with actions of the Hamiltonian form (4.5). Instead we use the Lagrangian formalism, in which we do not consider the fields as living on a particular spatial slice Σ at each moment in time, but rather as living on all of spacetime at once. Then the action is simply the integral over spacetime $\mathcal{M}$ of some functional of the fields ϕ^a and their derivatives:

$$S = \int_{\mathcal{M}} L(\phi)\,\mathrm{d}^D x\,. \tag{4.7}$$

Here we are writing the dependence on the fields as $L = L(\phi)$, but this is really shorthand for $L = L(\phi, \partial\phi, \partial^2\phi, \dots)$.

It is usually the case that we can switch between the Lagrangian and canonical Hamiltonian points of view. As an example, consider the Lagrangian of a scalar field in Minkowski spacetime:

$$L(\phi) = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2. \tag{4.8}$$

To move to the canonical setting we need to pick a notion of time, and use it to slice spacetime $\mathcal{M}$ into spacelike surfaces Σ of constant time. Luckily in this case the choice is obvious – let's just use Minkowski time t, and $\Sigma = \mathbb{R}^{D-1}$. At a given time t, and point in space $(x_1, \ldots, x_{D-1}) \in \Sigma$, we define

$$\varphi(x_i) = \phi(t, x_i). \tag{4.9}$$

To get the momentum conjugate to φ, we take the Euler-Lagrange derivative of $L(\phi)$ with respect to its time derivative:

$$\pi(x_i) = \frac{\delta L}{\delta \partial_t \phi} = \partial_t \phi(t, x_i) = \dot{\varphi}(x_i). \tag{4.10}$$

Then the Hamiltonian is defined with a Legendre transform

$$\mathcal{H}(\varphi, \pi) = \pi\dot{\varphi} - L = \frac{1}{2}\pi^2 + \frac{1}{2}\delta^{ij}\partial_i\varphi\partial_j\varphi + \frac{1}{2}m^2\varphi^2. \tag{4.11}$$

It's simple to verify that the resulting Hamiltonian action is equivalent to the original Lagrangian one.

For the scalar field, this was simple enough to do. It can be done more generally, for more complicated theories, but there can be two problems with this.

First, its not usually so obvious which is the best time slicing to use. In this case, we used Minkowski time, and the reason this worked nicely is because of the underlying Poincaré symmetry. But there is also Lorentz symmetry, so we could have done a Lorentz boost, and this would lead to a different, but equally valid time slicing. In general, however, there will be no Poincaré symmetry. The choice of time slicing will be completely arbitrary, since there will be no symmetry principles which we can use to guide this choice.

Second, in many cases the Lagrangian action is much simpler than the Hamiltonian one. For example, consider the Einstein-Hilbert action that governs general relativity:

$$S = \frac{1}{16\pi G}\int_{\mathcal{M}} R\sqrt{-g}\,\mathrm{d}^D x\,. \tag{4.12}$$

There is only one field – the metric $g_{\mu\nu}$. If you follow all of the steps necessary to get to a canonical Hamiltonian action, you have to do a lot of messy algebra, and what you end up with is called the ADM (Arnowitt-Deser-Misner) action. You have to choose an (arbitrary) slicing of spacetime, and a division of the coordinates into space and time coordinates. Then you decompose the metric into its timelike and spacelike directions, which produces several fields – the induced metric h_{ij} on each spacelike surface, as well as the other components of the metric which are known as the lapse N and shift N^i.

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -N^2\,\mathrm{d}t^2 + h_{ij}(\mathrm{d}x^i + N^i\,\mathrm{d}t)(\mathrm{d}x^j + N^j\,\mathrm{d}t). \tag{4.13}$$

After working it through, you can show that the momenta conjugate to the lapse and shift vanish (these are primary constraints),

$$\frac{\delta L}{\delta \partial_t N} = \frac{\delta L}{\delta \partial_t N^i} = 0, \tag{4.14}$$

while the momentum conjugate to the spatial metric is

$$\pi^{ij} = \frac{\delta L}{\delta \partial_t h_{ij}} = \sqrt{h}(K^{ij} - K h^{ij}), \tag{4.15}$$

where

$$K_{ij} = \frac{1}{2N}\Big(\partial_t h_{ij} - D_i N_j - D_j N_i\Big) \tag{4.16}$$

is the extrinsic curvature of h_{ij}. Here we are raising and lowering $ij\ldots$ indices with h_{ij} and its inverse h^{ij}, and D_i is the covariant derivative associated with h_{ij}. In terms of these variables, the action is

$$S = \frac{1}{16\pi G}\int \mathrm{d}t\, \mathrm{d}^{D-1}x\, N\sqrt{h}\Big(R_h - K_{ij}K^{ij} + K^2\Big), \tag{4.17}$$

where R_h is the Ricci scalar of the spacelike metric. One now has to convert this into Hamiltonian form – we won't go into more of the details of this, but the point is that it is already becoming quite complex, especially compared to the simplicity of (4.12). Things become even more complicated when we add matter to the action.

These two drawbacks to the canonical approach are not unrelated. They both stem from there being not enough (or too much) symmetry. In the case of the first drawback, this makes it difficult to pick a preferred time-slicing. And in the second, it means that the physical fields are much easier to understand from the Lagrangian point of view, which emphasises the role played by the full structure of spacetime.

Fortunately, there is an alternative to the canonical Hamiltonian framework. This is the covariant phase space formalism, and we will learn about it over the next few weeks.

4.1 Lagrangian as a top form

Let us use the symbol ϕ to denote all of the dynamical fields in spacetime. This can include the metric $g_{\mu\nu}$ in gravity, as well as any matter fields, and anything else that might be evolving.

Up to now, we have been viewing the Lagrangian as a function on spacetime, and we have been defining the action as the integral of this function over spacetime. However, to do this integration, we need to pick a volume form, and in theories where the metric is dynamical (such as gravity), the volume form can vary. For this reason, it is actually more useful to define the Lagrangian to be a top form $L = L(\phi)$ on spacetime. Then we define the action as just the integral of this top form:

$$S = \int_{\mathcal{M}} L(\phi). \tag{4.18}$$

In terms of the Lagrangian function, we have $L = L_{\text{function}}\sqrt{-g}\,\mathrm{d}^D x$. In general at a point in spacetime $L(\phi)$ will depend on the fields ϕ and finitely many of their derivatives at that point.

We get the equations of motion by varying the fields ϕ. The Lagrangian form depends on ϕ and several of its derivatives, so after a variation $\phi \to \phi + \delta\phi$ we will have (at linear order in $\delta\phi$)

$$\delta L = L(\phi+\delta\phi) - L(\phi) = L_{(0)}(\phi)\cdot\delta\phi + L^{\mu}_{(1)}(\phi)\cdot\partial_\mu\delta\phi + \cdots + L^{\mu\ldots\nu}_{(k)}(\phi)\cdot\underbrace{\partial_\mu\ldots\partial_\nu}_{k}\delta\phi, \tag{4.19}$$

for some forms $L^{\mu\cdots}_{(i)}(\phi)$ that depend on the fields and their derivatives. Here we are using $\cdot$ to denote a summation over all fields (previously we used an index $_a$ and $\sum_a$). We can use the product rule to massage this into a more useful equation. Since we are dealing with differential forms, it is most convenient to use Lie derivatives. For example, we have[1]

$$L^{\mu}_{(1)} \cdot \partial_\mu \delta\phi = L^{\mu}_{(1)} \cdot \mathcal{L}_{\frac{\partial}{\partial x^\mu}} \delta\phi = \mathcal{L}_{\frac{\partial}{\partial x^\mu}} (L^{\mu}_{(1)} \cdot \delta\phi) - (\mathcal{L}_{\frac{\partial}{\partial x^\mu}} L^{\mu}_{(1)}) \cdot \delta\phi. \tag{4.20}$$

Now we can use the fact that the Lie derivative of a top form is an exact form. This can be seen from Cartan's magic formula for the Lie derivative of a differential form

$$\mathcal{L}_\xi \omega = \mathrm{d}(\iota_\xi \omega) - \iota_\xi \,\mathrm{d}\omega\,; \tag{4.21}$$

when ω is a top form we have $\mathrm{d}\omega = 0$ so the second term vanishes. Thus we have

$$L^{\mu}_{(1)} \cdot \partial_\mu \delta\phi = \mathrm{d}\Big(\underbrace{\iota_{\frac{\partial}{\partial x^\mu}} \big(L^{\mu}_{(1)} \cdot \delta\phi \big)}_{=\theta_{(1)}} \Big) - (\mathcal{L}_{\frac{\partial}{\partial x^\mu}} L^{\mu}_{(1)}) \cdot \delta\phi. \tag{4.22}$$

We can do this application of the product rule twice to obtain

$$L^{\mu\nu}_{(2)} \cdot \partial_\mu \partial_\nu \delta\phi = \mathrm{d}\Big(\underbrace{\iota_{\frac{\partial}{\partial x^\mu}} \big(L^{\mu\nu}_{(2)} \cdot \partial_\nu \delta\phi \big) - \iota_{\frac{\partial}{\partial x^\nu}} \big((\mathcal{L}_{\frac{\partial}{\partial x^\mu}} L^{\mu\nu}_{(2)}) \cdot \delta\phi \big)}_{=\theta_{(2)}} \Big) + (\mathcal{L}_{\frac{\partial}{\partial x^\nu}} \mathcal{L}_{\frac{\partial}{\partial x^\mu}} L^{\mu\nu}_{(2)}) \cdot \delta\phi. \tag{4.23}$$

In fact, we can do this for every term in (4.19), and will obtain

$$L^{\mu\ldots\nu}_{(i)} \cdot \partial_\mu \ldots \partial_\nu \delta\phi = \mathrm{d}\theta_{(i)}(\phi, \delta\phi) + (-1)^i (\mathcal{L}_{\frac{\partial}{\partial x^\nu}} \ldots \mathcal{L}_{\frac{\partial}{\partial x^\mu}} L^{\mu\ldots\nu}_{(i)}) \cdot \delta\phi, \tag{4.24}$$

for some $(D-1)$-forms $\theta_{(i)}$ that depend on ϕ, $\delta\phi$ and their derivatives. Substituting this into (4.19), we end up with

$$\delta L = E(\phi) \cdot \delta\phi + \mathrm{d}\theta(\phi, \delta\phi)\,, \tag{4.25}$$

where $\theta = \theta_{(1)} + \theta_{(2)} + \cdots + \theta_{(k)}$, and

$$E(\phi) = L_{(0)}(\phi) - \mathcal{L}_{\frac{\partial}{\partial x^\mu}} L^{\mu}_{(1)}(\phi) + \cdots + (-1)^k \mathcal{L}_{\frac{\partial}{\partial x^\nu}} \ldots \mathcal{L}_{\frac{\partial}{\partial x^\mu}} L^{\mu\ldots\nu}_{(k)}(\phi). \tag{4.26}$$

This is the form version of the Euler-Lagrange derivative of L, and it gives the equations of motion $E(\phi) = 0$. To see this, we can note that using Stokes' theorem the variation of the action is

$$\delta S = \delta\Big(\int_{\mathcal{M}} L \Big) = \int_{\mathcal{M}} \delta L = \int_{\mathcal{M}} E(\phi) \cdot \delta\phi + \int_{\partial\mathcal{M}} \theta(\phi, \delta\phi). \tag{4.27}$$

Ignoring the boundary term, we see that $\delta S = 0$ for arbitrary $\delta\phi$ implies that $E(\phi) = 0$, as expected.

Equation (4.25) is very important, and we will return to it often. In the covariant phase space formalism, the form θ is just as important as the equations of motion.

Note that although it is useful to have the general formula (4.26) for the equations of motion, in practice for a given action it is often simpler to not use it. Instead one just directly manipulates δL into the form in (4.25). Similarly, although it is possible to write down a general formula for θ, in practice it is usually easier to just obtain it directly.

[1] Here we are assuming ϕ transforms like a scalar field, but similar results apply for other types of field.

For example, consider again the Lagrangian of a scalar field.

$$L = -\frac{1}{2}\Big(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\Big)\,\mathrm{d}^D x\,. \tag{4.28}$$

Varying $\phi \to \phi + \delta\phi$, we can write the variation of the Lagrangian as

$$\delta L = \Big(\eta^{\mu\nu}\partial_\mu\partial_\nu\phi - m^2\phi\Big)\delta\phi\,\mathrm{d}^D x - \partial_\mu(\delta\phi\,\eta^{\mu\nu}\partial_\nu\phi)\,\mathrm{d}^D x\,. \tag{4.29}$$

We can write the latter term as $\mathrm{d}(\delta\phi * \mathrm{d}\phi)$, so for the scalar field we have

$$E = \Big(\eta^{\mu\nu}\partial_\mu\partial_\nu\phi - m^2\phi\Big)\,\mathrm{d}^D x\,, \qquad \theta = \delta\phi * \mathrm{d}\phi\,. \tag{4.30}$$

4.2 Boundary action and boundary conditions

When we obtained the equations of motion, we ignored the effects of the boundary contribution in (4.27). Let's now be a bit more careful about this.

In general, the action will have an additional contributions from the boundary of spacetime. These are called boundary terms, or the boundary action. Let us write

$$S = \int_{\mathcal{M}} L(\phi) + \int_{\partial\mathcal{M}} l(\phi). \tag{4.31}$$

In this equation, $l(\phi)$ is a Lagrangian $(D-1)$-form for the boundary. In order to distinguish between the two Lagrangian forms, we will sometimes refer to L as the bulk Lagrangian, and l as the boundary Lagrangian.

With the boundary term, the variation of the action may be written

$$\delta S = \int_{\mathcal{M}} E(\phi)\cdot\delta\phi + \int_{\partial\mathcal{M}} \Big(\theta(\phi,\delta\phi) + \delta l(\phi)\Big). \tag{4.32}$$

So, the boundary Lagrangian does not influence the equations of motion. However, it is clear that, even if the equations of motion $E = 0$ are satisfied, it may still be for certain field variations $\delta\phi$ that we have $\delta S \neq 0$. In particular, these field variations are non-trivial at the boundary $\partial\mathcal{M}$. This means that the equations of motion are not enough by themselves to ensure that the fields extremise the action.

To fix this, it is necessary in general to impose *boundary conditions*, i.e. conditions on the behaviour of the fields ϕ near the boundary $\partial\mathcal{M}$. These boundary conditions must be chosen such that they ensure the boundary contribution to δS vanishes.

The boundary conditions and boundary action are very closely linked, and must be compatible with one another. This is best demonstrated with an example, so consider again the scalar field, and let us suppose the boundary Lagrangian vanishes, $l = 0$. On-shell (i.e. when the equations of motion are satisfied), we have

$$\delta S = \int_{\partial\mathcal{M}} \theta = \int_{\partial\mathcal{M}} \delta\phi * \mathrm{d}\phi\,. \tag{4.33}$$

It is clear that if we impose $\delta\phi|_{\partial\mathcal{M}} = 0$ (i.e. that the value of ϕ on the boundary is fixed), we will have $\delta S = 0$. So with this boundary condition, the equations of motion are sufficient to extremise the action. Suppose instead we had a non-vanishing boundary Lagrangian $l = -\phi * \mathrm{d}\phi$. Then on-shell we would have

$$\delta S = \int_{\partial\mathcal{M}} (\theta + \delta l) = -\int_{\partial\mathcal{M}} \phi * \mathrm{d}\delta\phi \,, \tag{4.34}$$

and in this case $\delta\phi|_{\partial\mathcal{M}} = 0$ is not sufficient to ensure $\delta S = 0$, because the pullback of $*\,\mathrm{d}\delta\phi$ to $\partial\mathcal{M}$ depends on the normal derivative of $\delta\phi$ at the boundary. However in this case we could impose the boundary condition that the normal derivative of ϕ to the boundary is fixed; then we would have $\delta S = 0$. But this boundary condition would not work when $l = 0$.

In gravity and gauge theory, these boundary contributions turn out to be quite important.

4.3 Dynamical and background fields

Sometimes there a theory will depend on external parameters that are non-dynamical. These can be very simple – for example they can be just coupling constants. But they can also be more complicated objects, such as a background metric. In any given setup, these parameters are completely fixed. Let us use the symbol γ to collectively refer to them. The action, and boundary/bulk Lagrangians depend on them as well as ϕ:

$$S(\phi;\gamma) = \int_{\mathcal{M}} L(\phi;\gamma) + \int_{\partial\mathcal{M}} l(\phi;\gamma). \tag{4.35}$$

We refer to ϕ as *dynamical* fields, and γ as *background* fields.

Varying ϕ but keeping γ fixed, it is clear that the equations of motion and various boundary contributions depend on γ,

$$\delta S = \int_{\mathcal{M}} E(\phi;\gamma)\cdot\delta\phi + \int_{\partial\mathcal{M}} \Big(\theta(\phi,\delta\phi;\gamma) + \delta l(\phi;\gamma)\Big). \tag{4.36}$$

On the other hand, because the background fields are fixed in any given setup, we can't extremise the action with respect to them, so they have no associated equations of motion. But we can still ask what might happen if we were to change them. In fact, asking this question allows us to define a wide class of quite familiar field-theoretical objects. If we keep the dynamical fields ϕ fixed, but vary the background fields $\gamma \to \gamma + \delta\gamma$, the action will in general change by

$$\delta S = \int_{\mathcal{M}} J(\phi;\gamma)\cdot\delta\gamma + \int_{\partial\mathcal{M}} \Big(j_{(0)}(\phi;\gamma)\cdot\delta\gamma + j_{(1)}(\phi;\gamma)\cdot\partial_n\delta\gamma + \cdots + j_{(p)}(\phi;\gamma)\cdot\partial_n^p\delta\gamma\Big) \tag{4.37}$$

for some forms $J, j_{(0)}, j_{(1)}, \ldots, j_{(p)}$, where ∂_n denotes a derivative normal to the boundary. These forms measure the response of the theory to a perturbation of the background fields, which in this context are sometimes called sources. The forms $J, j_{(0)}, \ldots$ are known as currents, and we say that the sources are coupled to the currents via the pairing arising from the above equation.

As an example, consider an ordinary field theory coupled to a background metric $g_{\mu\nu}$, and a background gauge potential A_μ, and suppose we vary these background fields away from the boundary while keeping the other fields fixed. Then we can write

$$\delta S = \int_{\mathcal{M}} \left(\frac{1}{2}T^{\mu\nu}\delta g_{\mu\nu} + J^\mu \delta A_\mu\right)\sqrt{-g}\,\mathrm{d}^D x \tag{4.38}$$

for some $T^{\mu\nu}$ and J^μ. These objects are well-known; $T^{\mu\nu}$ is called the energy-momentum tensor, and J^μ is the electromagnetic current.

4.4 Covariance and general covariance

The primary theories of interest in this course will be *covariant* ones. It is probably about time that we defined what we mean by covariant.

Recall that the set of all diffeomorphisms which act on spacetime $\mathcal{M}$ forms a group $\mathrm{Diff}(\mathcal{M})$. In short, a covariant theory is one in which this group has an action on all of the fields (dynamical and background), under which the equations of motion are unchanged. To be more precise, let $f : \mathcal{M} \to \mathcal{M}$ be a diffeomorphism, under which $\phi \to \phi^f$ and $\gamma \to \gamma^f$. Then a covariant theory is one for which

$$E(\phi^f;\gamma^f) = 0 \iff E(\phi;\gamma) = 0 \tag{4.39}$$

holds for all f.

The simplest way to construct covariant theories is to use standard tensor calculus operations to construct the Lagrangian form $L(\phi;\gamma)$ out of the fields $\phi;\gamma$. This then will guarantee that the form $E(\phi;\gamma)$ will transform like a tensor under the action of $\phi \to \phi^f, \gamma \to \gamma^f$. As a consequence, the equations of motion $E = 0$ are tensorial, and so will automatically satisfy the requirement of invariance (4.39).

Almost all field theories can be written in a covariant manner. This can be done by adding sufficiently many new background fields. To get back the original theory one just then sets these new background fields to their original values. For example, the scalar field on Minkowski space is not covariant. But we can make it covariant by promoting the metric $\eta_{\mu\nu}$ to a general background metric $g_{\mu\nu}$. To get back the original theory we just set $g_{\mu\nu} = \eta_{\mu\nu}$. The covariant phase space formalism will turn out to be most useful when applied to covariant field theories. So from now on, we will assume that we are dealing with theories that have been made covariant.

On the other hand, there is a much stronger type of covariance that certainly does not hold for a general theory. This is *general covariance*. A theory is generally covariant if the equations of motion are unchanged when we apply a general diffeomorphism to the dynamical fields, $\phi \to \phi^f$, *but leave the background fields unchanged*. These kinds of diffeomorphisms are known as 'active' diffeomorphisms, whereas those that act on both the dynamical and background fields are known as 'passive' diffeomorphisms.

General covariance is one of the key properties that makes theories of gravity special. It strongly restricts the types of background field that can be part of a theory. To see this, suppose we have a theory of fields ϕ and background fields γ that we have written in a covariant way, and that moreover the theory is generally covariant. Covariance implies invariance under passive diffeomorphisms $\phi \to \phi^f$, $\gamma \to \gamma^f$, while general covariance implies invariance under active diffeomorphisms $\phi \to \phi^f$, $\gamma \to \gamma$. A covariant and generally covariant theory should therefore be also invariant under the third type of diffeomorphims $\phi \to \phi$, $\gamma \to \gamma^f$. If we consider an infinitesimal diffeomorphism parametrised by a vector field ξ, then this corresponds to the variation $\delta\phi = 0$, $\delta\gamma = \mathcal{L}_\xi \gamma$, and substituting this in to (4.37) we see that

$$\delta S = \int_{\mathcal{M}} J(\phi;\gamma) \cdot \mathcal{L}_\xi \gamma + \text{boundary terms.} \tag{4.40}$$

For the equations of motion to be invariant under this transformation, the integrand above needs to be independent of ϕ. This essentially leads to three possibilities:

- $\mathcal{L}_\xi \gamma = 0$. The only kind of γ that can satisfy this equation for all ξ are constant scalar fields, a.k.a. coupling constants.
- $J(\phi; \gamma)$ is independent of ϕ. Since J is essentially the derivative of the Lagrangian with respect to γ, this means that ϕ and γ are completely decoupled.
- γ only contributes at the boundary, so the bulk integral above automatically vanishes.

We can ignore any background fields which are decoupled from the dynamical fields, since these have no physical role to play. Thus, there are only two types of background field in generally covariant theories: coupling constants, and boundary sources.

This is quite a strong statement. It says that any non-trivial field in the bulk must be dynamical. This is why the principle of general covariance, plus any notion of spacetime distance, leads automatically to a theory of dynamical geometry – since any bulk metric in a generally covariant theory must be dynamical.

Let us write down the most general action for a generally covariant theory. We will count coupling constants separately, and not include them in the set of background fields γ. Thus, the action takes the form

$$S = \int_{\mathcal{M}} L(\phi) + \int_{\mathcal{M}} l(\phi; \gamma) \tag{4.41}$$

for some set of dynamical fields ϕ and boundary sources γ. Also, the linearised variation of the action under $\phi \to \phi + \delta\phi$ and $\gamma \to \gamma + \delta\gamma$ takes the form

$$\delta S = \int_{\mathcal{M}} E(\phi) \cdot \delta\phi + \int_{\partial\mathcal{M}} \Big(\theta(\phi, \delta\phi) + \delta l(\phi; \gamma)\Big). \tag{4.42}$$

The boundary currents $j_{(0)}, j_{(1)}, \ldots$ may be extracted from the form of δl.

As a final point, let us note that there is a deep connection between boundary conditions and boundary sources. To see this, it is helpful to view the different boundary conditions that one could impose as the values taken by a background field. For example, in the case of the scalar field with action

$$S = -\frac{1}{2} \int_{\mathcal{M}} \mathrm{d}^D x \left(\eta^{\mu\nu} \partial_\mu\phi \partial_\nu\phi + m^2\phi^2\right), \tag{4.43}$$

the on-shell variation of the action was

$$\delta S = \int_{\partial\mathcal{M}} \delta\phi * \mathrm{d}\phi \,, \tag{4.44}$$

and we noted that for this to vanish we needed to set the boundary condition $\delta\phi|_{\partial\mathcal{M}} = 0$, i.e. we need to fix the value of ϕ on the boundary. But we didn't specify what we would fix the value of ϕ to be. We can view this fixed boundary value of ϕ as a background field $\gamma = \phi|_{\partial\mathcal{M}}$. This field γ lives on the boundary, so we can view it as a boundary source. Moreover, we can use the above to find the boundary current to which it is coupled:

$$\delta S = \int_{\partial\mathcal{M}} \delta\gamma * \mathrm{d}\phi = -\int_{\partial\mathcal{M}} \epsilon \, \delta\gamma \, \partial_n\phi, \tag{4.45}$$

where ϵ is the induced volume form on $\partial\mathcal{M}$. Thus, γ sources $\partial_n\phi$.

(This is one of the main conceptual ingredients that goes into the holographic principle, where the background fields in a $(D-1)$-dimensional theory set the boundary conditions of a D-dimensional theory.)