2.3 Time Reparametrization Invariance

We have seen above how constraints arise in a classical mechanical system, standard techniques to deal with them, and a powerful geometric reformulation in terms of presymplectic manifolds. But there exists a specific class of constrained systems which can be understood as being fundamentally distinct. These are dynamically constrained systems, with a vanishing canonical Hamiltonian and a time reparametrization invariant action. Dynamically constrained systems are inherently different from usual symmetry constrained systems in general, because they are characterised by a temporal gauge symmetry. In other words, the physical evolution parameter ("time") of Newtonian mechanics is no longer an absolute background structure, now being included as one of the dynamical variables q^i. Moreover, the 'evolution' parameters characterising the trajectories of dynamically constrained systems are now gauge, since these trajectories are gauge orbits generated by a first class Hamiltonian constraint. Time is what parametrizes a trajectory, and in this sense "time is gauge" in dynamically constrained systems. A prime example is gravity.

Let us start with a simple illustrative example highlighting the main features of the presymplectic formulation presented above, and bringing to the fore those aspects that are especially useful for generally covariant systems. We will then move on to a discussion and examples of time reparametrization invariant systems. For simplicity, we will restrict to systems with a single first class constraint (like in section 2.2.2 above, cf. figure 2).

2.3.1 Presymplectic mechanics: An example

For unconstrained systems, we saw in section 2.1.1 that the standard Hamilton's equations of motion can be written in a compact geometric form as,

$$\iota_{X_H}\omega = -\mathrm{d}H \tag{2.54}$$

where H is a (smooth) Hamiltonian function on $T^*\mathcal{Q}$, $\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i$ is the symplectic 2-form on $T^*\mathcal{Q}$, and X_H is the corresponding Hamiltonian vector field with coordinate expression given by (2.14). While for constrained systems, we saw in section 2.2.2 that the equations of motion can be written in a similarly compact geometric form on the presymplectic constraint surface, as

$$\iota_X\omega_0 = 0 \tag{2.55}$$

where ω_0 is the presymplectic 2-form on the constraint surface Σ_0 s.t. $(FL)^*\omega_0 = \omega_L$, and X is a vector field on Σ_0. Recall that each such *null* vector field (i.e. one that satisfies the above equation) is in 1-1 correspondence with a first class constraint as discussed above in section 2.2.2. Notice that we can drop the subscript and denote by $\Sigma \equiv \Sigma_0 = \Sigma_f$, the final constraint surface since we are only considering a single first class constraint.

In particular equation (2.55) encodes the complete physics of the system in a covariant manner. For unconstrained systems, this presymplectic reformulation can simply be seen as an alternative formulation that is manifestly covariant. But for constrained systems, such a covariant formulation is fundamentally crucial. To see this, let us consider a simple example.

Example. To illustrate how equation (2.55) encodes the complete dynamical information about the system, let us consider the following. Let Σ be a 3-dim, presymplectic manifold, with the presymplectic 2-form given by

$$\omega_0 := \mathrm{d}z \wedge \mathrm{d}x - (x\mathrm{d}x + z\mathrm{d}z) \wedge \mathrm{d}y \tag{2.56}$$

in local coordinates $(x, y, z) \in \Sigma$. The equation of motion is given by (2.55), i.e.

$$\begin{aligned} 0 &= \iota_X \omega_0 && (2.57)\\ &= (X^z + xX^y)\mathrm{d}x - (xX^x + zX^z)\mathrm{d}y + (zX^y - X^x)\mathrm{d}z && (2.58) \end{aligned}$$

where $X = X^x\partial_x + X^y\partial_y + X^z\partial_z \in \mathfrak{X}(\Sigma)$, and we have used the identity $\iota_X\omega = \omega_{\mu\nu}X^\mu \mathrm{d}a^\nu - \omega_{\mu\nu}X^\nu \mathrm{d}a^\mu$, for an arbitrary 2-form ω and vector field field X on Σ (with $\mu, \nu = 1, 2, 3$ and local coordinates $a^\mu = (x, y, z)$). Thus, we have

$$X^z + xX^y = 0\,, \qquad xX^x + zX^z = 0\,, \qquad zX^y - X^x = 0\,. \tag{2.59}$$

This implies that,

$$X = \frac{X^x}{z}\left(z\partial_x - x\partial_z + \partial_y\right)\,. \tag{2.60}$$

Notice that if $X \in \ker\omega_0$, then $kX + \ell Y \in \ker\omega_0$, where $Y \in \ker\omega_0$ and $k, \ell \in \mathbb{R}$. Thus, rescaling a null vector field by an arbitrary function does not change the physical equations of motion. In other words, the null orbits are invariant under arbitrary reparametrizations[16]. For the present example then, the null vector field is equivalently given by,

$$X = z\partial_x - x\partial_z + \partial_y \equiv X_H + \partial_y\,. \tag{2.61}$$

Notice that X_H is defined on the subspace $\{(x, z)\}|_{\mathrm{const}\,y}$ of Σ. Further, $\omega_0|_{\mathrm{const}\,y} = \mathrm{d}z \wedge \mathrm{d}x$. This defines a symplectic leaf $\mathcal{P}_y \subset \Sigma$ with symplectic 2-form $\tilde{\omega} = \omega_0|_{\mathrm{const}\,y}$.

We can then ask: which function $H \in \mathcal{F}(\mathcal{P}_y)$ is the above X_H the Hamiltonian vector field of? To answer this, we must solve the defining equation of a Hamiltonian vector field, namely

$$\iota_{X_H}\tilde{\omega} = -\mathrm{d}H\,. \tag{2.62}$$

Expanding each side in local coordinates, we have

$$\begin{aligned} \iota_{X_H}\tilde{\omega} &= -z\mathrm{d}z - x\mathrm{d}x\,, && (2.63)\\ -\mathrm{d}H &= -\partial_x H\mathrm{d}x - \partial_z H\mathrm{d}z\,. && (2.64) \end{aligned}$$

Thus,

$$\partial_x H(x, z) = x\,, \qquad \partial_z H(x, z) = z \tag{2.65}$$

which can be identified as the standard Hamilton's equations of motion for a free simple harmonic oscillator (SHO), with the Hamiltonian

$$H(x, z) = \frac{1}{2}(x^2 + z^2) + \mathrm{const}\,. \tag{2.66}$$

[16] For Hamiltonian constraints, this is in fact the statement that physical motions are time reparametrization invariant, as we will see later below.

Therefore, we can explicitly see that complete dynamical information of the SHO is encoded in its presymplectic equation of motion (2.57).

Further, notice that we can now identify the variable z as the canonical conjugate momentum of position x. The fact that variables z and x are canonical conjugates is suggested also by the definition of ω_0 in (2.56). We can now also understand the variable y as "time" (here, Newtonian). The fact that we did not need to interpret any of these variables as such *a priori* is an important feature of the formalism, especially in the context of generally covariant systems. Even more importantly, the equations of motion (2.57) did not need to be written in terms of an evolution with respect to a distinguished time variable, nor do they automatically always identify any such preferred variable. In other words, at the level of the constraint surface (which is where all physical information resides), all variables are mixed up. None is required to be identified as time or space a priori[17]. This is how the presymplectic formulation is covariant while also being geometric in nature, and why it is very useful in the context of gravity.

Lastly, let us relabel $p \equiv z$ and $t \equiv y$, to regain familiarity with the standard equations for a free SHO. The equations of motion (2.65) can be easily solved to give the following well known solutions,

$$x(t) = A \sin(t + \theta)\,, \qquad p(t) = A \cos(t + \theta) \tag{2.67}$$

where amplitude A and phase θ are arbitrary integration constants. These equations are graphs of physical motions, i.e. of the gauge-fixed integral curves of the null vector field of ω_0 (cf. orbits in red, in Figure 2). Each null orbit is labelled by a distinct pair (A, θ). Therefore, elements of the equivalence class of orbits are precisely (A, θ), the initial conditions for a free SHO. In other words, the physical phase space $\mathcal{P}_{\text{phys}}$ (2.51) of the SHO, is coordinatised by $A^2 = x^2 + p^2$, and $\theta = \tan^{-1}(x/p) - t$, with symplectic structure $\omega_{\text{phys}} = A\mathrm{d}A \wedge \mathrm{d}\theta$.

2.3.2 Homogeneous Lagrangian

Consider an action,

$$S[q^I] = \int_{\tau_1}^{\tau_2} d\tau \; L(q^I(\tau), \dot{q}^I(\tau)) \tag{2.68}$$

where dot refers to derivative with respect to τ, and the Lagrangian satisfies the following property,

$$L(q^I, \kappa \dot{q}^I) = \kappa L(q^I, \dot{q}^I) \tag{2.69}$$

with $\kappa(\tau)$ being an arbitrary function. A Lagrangian satisfying the property (2.69) above is said to be *homogeneous* in the velocities with degree 1. This is an important property, because it relates to two other important properties.

1. **Reparametrization invariance.** Equation (2.69) implies that the corresponding action $S[q^I(\tau)]$ is invariant under reparametrizations $\tau \mapsto \tilde{\tau} = \tilde{\tau}(\tau)$, where the inverse

[17] Nevertheless, if an absolute (or background) structure does exist (like the Newtonian time y in the above example), then it can eventually be identified and determined as such to be a background structure in disguise at the presymplectic level, from the form of the constraint. See [18, 19] for some insightful discussions.

$\tau(\tilde{\tau})$ exists. This can be easily seen as follows:

$$\begin{aligned}
S[q^I(\tilde{\tau})] &= \int_{\tilde{\tau}_1}^{\tilde{\tau}_2} d\tilde{\tau}\; L\left(q^I(\tilde{\tau}), \frac{dq^I(\tilde{\tau})}{d\tilde{\tau}}\right) \\
&= \int_{\tau_1}^{\tau_2} d\tau\, \frac{d\tilde{\tau}}{d\tau}\; L\left(q^I(\tilde{\tau}(\tau)), \frac{dq^I(\tilde{\tau}(\tau))}{d\tau}\frac{d\tau}{d\tilde{\tau}}\right) \\
&= \int_{\tau_1}^{\tau_2} d\tau\; L\left(q^I(\tau), \frac{dq^I(\tau)}{d\tau}\right) = S[q^I(\tau)]
\end{aligned} \tag{2.70}$$

where, without loss of generality, we have assumed $d\tilde{\tau}/d\tau > 0$ when replacing the limits of the integral. This feature of reparametrization invariance of the dynamics is what underlies the loss of background structures, specifically as being unphysical and unobservable, in generally covariant systems. For example, if τ is a time variable (in some specific sense determined by the precise context under consideration), then the invariance of the action under its reparametrizations means that this variable is not observable, and evolution in it is gauge.

2. **Zero canonical Hamiltonian.** Equation (2.69) is equivalent to the property that the canonical Hamiltonian, $H = p\dot{q} - L$, vanishes. The statement, (2.69) $\Rightarrow H = 0$, can be shown as follows:

$$\begin{aligned}
H &= \frac{\partial L(q^J, \dot{q}^J)}{\partial \dot{q}^I}\dot{q}^I - L(q^J, \dot{q}^J) \\
&= \frac{1}{\kappa}\frac{\partial L(q^J, \kappa\dot{q}^J)}{\partial \kappa\dot{q}^I}\kappa\dot{q}^I - \frac{1}{\kappa}L(q^J, \kappa\dot{q}^J) = \frac{1}{\kappa}H\,.
\end{aligned} \tag{2.71}$$

Since this holds for arbitrary $\kappa(\tau)$, this implies that $H = 0$. The converse statement can also be easily shown as follows: notice that,

$$H = 0 \quad \Rightarrow \quad \frac{\partial L(q^J, \kappa\dot{q}^J)}{\partial \kappa\dot{q}^I}\kappa\dot{q}^I = L(q^J, \kappa\dot{q}^J)\,. \tag{2.72}$$

Then, simplifying the left hand side, we have

$$\begin{aligned}
\frac{\partial L(q^J, \kappa\dot{q}^J)}{\partial \kappa\dot{q}^I}\kappa\dot{q}^I &= \kappa\frac{\partial L(q^J, \dot{q}^J)}{\partial \dot{q}^I}\dot{q}^I \\
&= \kappa p_I \dot{q}^I = \kappa L(q^J, \dot{q}^J)
\end{aligned} \tag{2.73}$$

where the last equality uses $H = 0$ in the Legendre transform. Comparing with the right hand side, we thus get (2.69).

This property of a zero canonical Hamiltonian is characteristic of dynamically (or, fully) constrained systems. In particular, notice that $H = 0$ implies that the extended Hamiltonian (2.47) is now a sum of only first class constraints.

A natural question that arises then is how the latter two properties, reparametrization invariance and vanishing canonical Hamiltonian, are mutually related. In all cases of practical interest, the two are equivalent, i.e. an action is reparametrization invariant if and only if its canonical Hamiltonian vanishes.[18]

2.3.3 Parametrized non-relativistic particle

We have already encountered a specific example of a parametrized non-relativistic particle, the free SHO discussed in section 2.3.1. Here we formalise the details for a generic non-relativistic particle, especially in light of the discussions above.

Consider a non-relativistic particle on configuration space $\mathcal{Q}$ coordinatized by (q^i). The action is given by,

$$S[q^i(t)] = \int_{t_1}^{t_2} dt\, L\left(q^i, \frac{dq^i}{dt}\right) \tag{2.74}$$

where Lagrangian $L \in \mathcal{F}(T\mathcal{Q})$. The central idea behind parametrizing a system is to raise all its variables to an equal footing, as canonical variables. Specifically here, this means that the variable t is now raised to being one of the configuration variables, in an extended configuration space $\mathcal{Q}_{\text{ex}} = \mathbb{R} \times \mathcal{Q} \ni (t, q^i) \equiv (q^I)$; and the variables now depend on a new parameter τ, i.e. $t(\tau), q^i(\tau)$. Consequently, the above action can be extended as,

$$\tilde{S}[t(\tau), q^i(\tau)] = \int_{\tau_1}^{\tau_2} d\tau\, \dot{t}\, L\left(q^i(\tau), \frac{\dot{q}^i}{\dot{t}}\right) \equiv \int_{\tau_1}^{\tau_2} d\tau\, \tilde{L}(q^i, \dot{q}^i, \dot{t}) \tag{2.75}$$

where dots denote derivatives with respect to τ, and $\tilde{L} \in \mathcal{F}(T\mathcal{Q}_{\text{ex}})$. Notice that $\tilde{L}$ is homogeneous in the velocities with degree 1, i.e. it satisfies equation (2.69). From the discussion in section 2.3.2 above, this implies that $\tilde{S}$ is invariant under arbitrary reparametrizations of τ, and that there exists an associated Hamiltonian constraint. This constraint is primary and first class, as will be evident below from the expression of the canonical momentum conjugate to t for the parametrized system.

Since now t is also a configuration variable in $\mathcal{Q}_{\text{ex}}$, there exists a corresponding conjugate momentum, given by

$$\begin{aligned} p_t = \frac{\partial \tilde{L}}{\partial \dot{t}} &= L(q^i, \dot{q}^i/\dot{t}) - \frac{\dot{q}^i}{\dot{t}} \frac{\partial L(q^i, \dot{q}^i/\dot{t})}{\partial(\dot{q}^i/\dot{t})} \\ &= L(q^i, dq^i/dt) - p(dq^i/dt) \\ &= -H(q^i, p_i)\,. \end{aligned} \tag{2.76}$$

That is, p_t, p_i and q^i are constrained to satisfy the equation (2.76). This gives us the Hamiltonian constraint,

$$\gamma = p_t + H \approx 0\,. \tag{2.77}$$

The canonical momenta conjugate to q^i are,

$$\tilde{p}_i = \frac{\partial \tilde{L}}{\partial \dot{q}^i} = \frac{\partial L(q^i, \dot{q}^i/\dot{t})}{\partial(\dot{q}^i/\dot{t})} = \frac{\partial L(q^i, dq^i/dt)}{\partial(dq^i/dt)} = p_i \tag{2.78}$$

[18] However, at least formally, it is possible for a system to be time reparametrization invariant but have a non-zero Hamiltonian. This is when the canonical variables q and p do not transform as scalars. We refer to [10] for details.

i.e. the standard momenta, conjugate to positions of the particle, are unchanged in the extended parametrized description. Then, the corresponding extended Hamiltonian, $H_{\rm ex} \in \mathcal{F}(T^*\mathcal{Q}_{\rm ex})$, is

$$H_{\rm ex} = p_i \dot{q}^i + p_t \dot{t} - \tilde{L} = \dot{t}\gamma\,, \tag{2.79}$$

in terms of which, the extended action can be rewritten as

$$\tilde{S}[t(\tau), q^i(\tau)] = \int_{\tau_1}^{\tau_2} d\tau\; p_i \dot{q}^i + p_t \dot{t} - N(\tau)\gamma \tag{2.80}$$

where,

$$N(\tau) = \frac{dt(\tau)}{d\tau}\,. \tag{2.81}$$

Function N is the so-called *lapse* function, which basically dictates how much "t-time" has lapsed with respect to the "τ-parameter time". It is an arbitrary function of τ, encoding the time reparametrization symmetry of the system. Further, it is the Lagrange multiplier imposing the constraint (2.77).

Now, $H_{\rm ex}$ generates evolution in the unphysical τ parameter. Therefore the corresponding trajectories are not physical motions. Physical information is instead encoded within gauge-invariant correlations between the variables $(t, q^i, p_i) \in \Sigma$. This requires us to first solve the equations of motion.

For example, for a free non-relativistic particle (mass = 1/2), the Hamiltonian constraint is $\gamma = p_t + p^2$. Solution to the equations of motion is,

$$t(\tau) = \tau + t_0\,, \qquad q(\tau) = 2p\tau + q_0 \tag{2.82}$$

for some arbitrary initial conditions (t_0, q_0). One question of operational relevance is: what is the position of the particle at time t_0? To answer questions like these in constrained systems, the idea is to compute coincidences of the variables in terms of gauge-invariant functions. Then, the question should be: what is the value of $q(\tau)$ when the value of $t(\tau)$ is t_0? In this manner of asking, we are removing the dependence on the gauge parameter τ. The function,

$$Q_{t_0}(\tau) = q(\tau) - 2p(\tau)(t(\tau) - t_0) \tag{2.83}$$

answers this question as: the particle is at position (value of) Q_{t_0} at time t_0. It can be checked that Q_{t_0} indeed weakly commutes with the Hamiltonian constraint, as required.

2.3.4 Free relativistic particle

Let us consider the action for a free relativistic particle,

$$S = -m \int_{s_1}^{s_2} ds = -m \int_{\tau_1}^{\tau_2} d\tau\, \sqrt{-\dot{x}_\mu \dot{x}^\mu}\,, \tag{2.84}$$

where m is the particle mass, ds is the proper time interval along the worldline, τ is an arbitrary parameter, and $\dot{x}^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$. Minkowski metric is $\eta = (-1, 1, 1, 1)$. The worldline is timelike (since $m > 0$), explaining the need for the minus sign under the square root. As in the previous example, it can be verified that the Lagrangian is indeed homogeneous in

$\dot{x}^\mu$ with degree 1. Thus, again we have that S is τ-reparametrization invariant and that there exists a Hamiltonian constraint. The constraint can be derived from the canonical momenta,

$$p_\mu = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}_\nu \dot{x}^\nu}} \,. \tag{2.85}$$

This implies the following Hamiltonian constraint,

$$\gamma = p^2 + m^2 \approx 0 \,. \tag{2.86}$$

As before, the action can be written in terms of this constraint as,

$$S = \int_{\tau_1}^{\tau_2} d\tau \; p_\mu \dot{x}^\mu - N\gamma \tag{2.87}$$

where the lapse function is,

$$N(\tau) = \frac{1}{2m}\frac{ds(\tau)}{d\tau} \,. \tag{2.88}$$

Lastly, again in order to derive possible observables like (2.83) above, we must first solve the equations of motion. The equations of motion are,

$$\dot{p}_\mu = -\frac{\partial N\gamma}{\partial x^\mu} = 0 \,, \qquad \dot{x}^\mu = \frac{\partial N\gamma}{\partial p_\mu} = 2Np^\mu \,. \tag{2.89}$$

Solution is,

$$p_\mu = \text{constant} \,, \qquad x^\mu = 2Np^\mu \Delta\tau \,. \tag{2.90}$$

Then, for example, the following gauge-invariant function,

$$X^i_{t_0} = x^i - \frac{p^i}{p^0}(x^0 - t_0) \tag{2.91}$$

can be used to answer the question: what are the coordinates x^i when coordinate x^0 takes the value t_0? But we know that locally there is no preferred Lorentz frame. In other words, say x^1 is as good as x^0 with respect to which we can describe the other variables. This means that different such gauge-invariant observables can be constructed, with respect to different x^μ's. Different such choices are known as different *deparametrizations* of the same system.

2.4 Further Reading

- Symplectic mechanics: [1–4]
- Theory of constraints: [5, 8–11]
- Examples, Reparametrization invariance: [10, 14, 18–21]

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