2.2 Constrained Hamiltonian Systems

Many interesting physical systems which admit a Lagrangian description are characterised by so-called *degenerate Lagrangians*, i.e. the Hessian matrix of the Lagrangian w.r.t. the velocities is degenerate. This is in fact the case of all fundamental interactions, including gravity. For such systems, the Legendre transformation is not invertible and, passing at the Hamiltonian description, this implies the existence of constraints providing functional relations between the canonical variables. A canonical formulation of the dynamics would then require to appropriately take into account the constraints. This can be done by following the so-called Dirac algorithm for constrained systems [8, 9], which we will briefly review in this section. Other references for the topic including excellent reviews are [10–13] (as well as [5, 6] for a modern geometric description including a Lagrangian counterpart of the constraint algorithm). We refer to them for those technical details that will be omitted here for brevity.

2.2.1 Singular Lagrangians and Dirac's Algorithm

The discussion of Sec. 2.1 was based on the assumption of the Lagrangian function to be regular, that is the associated Hessian matrix $\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$ was assumed to be non singular. Such a regularity assumption has profound consequences both on the description of the dynamics on $T\mathcal{Q}$ and on $T^*\mathcal{Q}$, as well as on the transition from one description to the other. Indeed, as we have seen at the beginning of the Chapter, in the case of regular Lagrangians, the Legendre map (2.10) provides a (local) diffeomorphism between $T\mathcal{Q}$ and $T^*\mathcal{Q}$ or in other words, each point $(q^i, \dot{q}^i) \in T\mathcal{Q}$ is mapped to a unique point $(q^i, p_i) \in T^*\mathcal{Q}$ and vice-versa. This means that we are able to invert the map FL to express all the velocities $\dot{q}^i$ (hence the accelerations $\ddot{q}^i$) in terms of the canonical momenta p_i and generalized coordinates q^i. This essentially encodes the equivalence between Lagrangian and Hamiltonian mechanics in the sense that we can visualize the dynamical trajectories either as integral curves of the Euler-Lagrange equations in $T\mathcal{Q}$ or as solutions of the Hamilton equations in $T^*\mathcal{Q}$.
Let us consider now the case of a degenerate Lagrangian, i.e. $\det\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right) = \det\left(\frac{\partial p_i}{\partial \dot{q}^j}\right) = 0$. In other words, now the Hessian matrix has less than maximum rank, namely

$$\text{rank}\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right) = K < N \; . \tag{2.28}$$

At the Lagrangian level, this means that the Euler-Lagrange equations are unable to uniquely determine the accelerations $\ddot{q}$ as functions of q and $\dot{q}$. At the canonical level, this implies that the q's and p's are not all independent as locally only K of the N momenta $p_i = \frac{\partial L}{\partial \dot{q}^i}$ can be inverted to express the velocities $\dot{q}^i$ in terms of q and p. Therefore, in this case, there are only $N + K$ independent phase space variables and the Legendre transformation identifies a $(N + K)$-dimensional subspace Σ_0 of the $2N$-dimensional phase space defined by $(N - K)$ functionally independent relations

$$\varphi_\alpha(q, p) = 0 \qquad , \qquad \alpha = 1, \ldots, N - K \; , \tag{2.29}$$

called *primary constraints*. Therefore, in the singular case, the Legendre map FL is not a diffeomorphism as its range is not the whole of $T^*\mathcal{Q}$ but only a $(N + K)$-dimensional submanifold Σ_0 of the phase space $\mathcal{P} = T^*\mathcal{Q}$, i.e.

$$\begin{aligned} &FL : T\mathcal{Q} \ni (q^i, \dot{q}^i) \longmapsto (q^i, p_i = \frac{\partial L}{\partial \dot{q}^i}) \in \Sigma_0 \subset T^*\mathcal{Q} \\ &\Sigma_0 = \Big\{ (q^i, p_i) \in T^*\mathcal{Q} \; \big| \; \varphi_\alpha(q,p) = 0 \; , \; \alpha = 1, \dots, N-K \Big\} \subset T^*\mathcal{Q} \; , \end{aligned} \tag{2.30}$$

the latter is called the *primary constraint submanifold*. Note that the Legendre transform still has the property that

$$\mathrm{d}(p_i \dot{q}^i - L) = \dot{q}^i \mathrm{d}p_i + p_i \mathrm{d}\dot{q}^i - \frac{\partial L}{\partial q^i}\mathrm{d}q^i - \frac{\partial L}{\partial \dot{q}^i}\mathrm{d}\dot{q}^i = \dot{q}^i \mathrm{d}p_i - \frac{\partial L}{\partial q^i}\mathrm{d}q^i \; , \tag{2.31}$$

i.e., q^i and p_i are the dynamical variables of the Hamiltonian formulation. However, the Hamiltonian now is not unique due to the presence of the constraints (2.29). Indeed, any function $f \in \mathcal{F}(T\mathcal{Q})$, when we try to express it in terms of phase-space variables, will appear as a function of the $(N+K)$ independent variables (the N configuration variables and the K independent momenta) as well as the $(N-K)$ unsolved velocities. Therefore, there is some freedom in the functional form in virtue of the choice of the unsolved velocities and the independent momenta. In particular, even if the canonical Hamiltonian defined as the Legendre transform of the Lagrangian restricted to the primary constraint surface Σ_0

$$H_0 := H(q,p)\big|_{\Sigma_0} = p_i \dot{q}^i - L(q,p)\big|_{\Sigma_0} \qquad (\text{s.t. } (FL)^* H_0 = \mathcal{E}_L) \tag{2.32}$$

does not depend on the (unsolved) velocities and so it can be considered as a function only of the q's and p's, any *total Hamiltonian* obtained by adding to it a linear combination of the primary constraints

$$H_T = H_0 + u^\alpha \varphi_\alpha \; , \tag{2.33}$$

would be on the same footing. The coefficients u^α are to be treated as Lagrange multipliers and are arbitrary functions of time (as well as of q and p). The inclusion of the primary constraints in the Hamiltonian makes the Legendre transformation invertible. The Hamiltonian EOMs obtained from (2.33) reads as

$$\begin{cases} \dot{q}^i = \frac{\partial H_0}{\partial p_i} + u^\alpha \frac{\partial \varphi_\alpha}{\partial p_i} \\ \dot{p}_i = -\frac{\partial H_0}{\partial q^i} - u^\alpha \frac{\partial \varphi_\alpha}{\partial q^i} \end{cases} \tag{2.34}$$

so that the time derivative of a generic phase space function $f(q,p)$ is now given by its Poisson bracket with the Hamiltonian (2.33)

$$\dot{f} = \{t, H_T\} = \{f, H_0\} + u^\alpha \{f, \varphi_\alpha\} + \{f, u^\alpha\}\varphi_\alpha = \{f, H_0\} + u^\alpha \{f, \varphi_\alpha\} \; . \tag{2.35}$$

Note that the constraints must be imposed only after Poisson brackets are computed. Following Dirac [9, 10], the latter property is usually denoted by a so-called *weak equality* $\approx$, which is an equality modulo the constraints, i.e. equality on the constraint hypersurface, and can be used only after all Poisson brackets have been evaluated.

In this sense, the (primary) constraint equations are understood as $\varphi_\alpha(q,p) \approx 0$, but $\{\varphi_\alpha, f\} \not\approx 0$ in general, so that we have the following *generalised EOM*

$$\dot{f} = \{t, H_T\} \approx \{f, H_0\} + u^\alpha \{f, \varphi_\alpha\} \; . \tag{2.36}$$

Remark 1. *The standard Poisson brackets* (2.22) *on* $\mathcal{P}$ *yield* $\{q^i, p_j\} = \delta^i_j$, $\{q^i, q^j\} = \{p_i, p_j\} = 0$ *which can be geometrically described in terms of a* Poisson bivector field *(i.e. a (2,0)-type skewsymmetric tensor)* Λ *such that*

$$\{f, g\} = \omega(X_f, X_g) = \Lambda(\mathrm{d}f, \mathrm{d}g) \ \ \forall f, g \in \mathcal{F}(\mathcal{P}) \qquad , \qquad \Lambda = \delta_{ij}\frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_j} \,. \tag{2.37}$$

Unlike differential forms which can be pulled-back to a submanifold, a (multi-)vector field cannot be "restricted" to a submanifold. From this point of view, the idea of Dirac can essentially be embodied in the statement: "if we cannot restrict the vector, then we enlarge the functions". This is achieved by the introduction of a new Hamiltonian function (2.33) *which "enlarges" the original Hamiltonian* H_0 *by additional terms taking into account the primary constraints which are zero on the (sub)manifold* Σ_0.

It should be stressed that (2.34) is the initial form of the Hamiltonian phase-space EOMs. To arrive at their final form, the theory proceeds with a step-by-step consistency analysis of constraints. Indeed, consistency of the constraints with Hamiltonian evolution implies the following stability conditions

$$0 \overset{!}{\approx} \dot{\varphi}_\alpha = \{\varphi_\alpha, H_T\} \approx \{\varphi_\alpha, H_0\} + u^\beta\{\varphi_\alpha, \varphi_\beta\} \,, \tag{2.38}$$

which geometrically amount to the requirement of dynamics to be tangent to the primary constraint surface and not carrying out of it, the latter as such remains stable under time evolution. Such consistency conditions can lead to the following four possibilities:

1) the conditions (2.38) are trivially satisfied (e.g. $0 = 0$), in which case the procedure ends here, all the u^β remain undetermined, and Eqs. (2.34) are the final form of the EOMs. Obviously, to look for solutions of such equations as the trajectories of the dynamics in Σ_0, we have to choose or to specify the unknown functions u in some way (*gauge fixing*);

2) the conditions (2.38) are never satisfied and the theory is inconsistent (*non physical pathological examples*)[9];

3) the conditions (2.38) impose restrictions on the u's;

4) the conditions (2.38) lead to relations that are independent of the u's thus yielding new constraints, say $\chi_m(q, p) \approx 0$. The new constraints generated in this way are called *secondary constraints*[10] and will in turn lead to new consistency conditions. The

[9] As an *ad hoc* example, consider Lagrangians of the form $L(q, \dot{q}) = Aq + B\dot{q}$, for some constants $A, B \neq 0$. As can be checked by direct computation, such Lagrangians are degenerate $\frac{\partial^2 L}{\partial q^2} = 0$ and the canonical momentum $p = \frac{\partial L}{\partial \dot{q}}$ yields the primary constraint $\varphi = p - B \approx 0$. The total Hamiltonian is given by $H_T = H_0 + u(p - B)$, with $H_0 = p\dot{q} - L = (p - B)\dot{q} - Aq$, and the constraint is not preserved by the dynamics as $\dot{\varphi} \approx \{\varphi, H_T\} \approx A\{q, p\} = A \neq 0$. The problem with such a Lagrangian is that it is not bounded, i.e., the associated action admits no extremal points thus resulting into inconsistent EOMs (EL equations $0 = -A \neq 0$).

[10] Note that for secondary constraints, one uses the EOMs, as opposed to primary constraints which instead are kinematical relations arising from the definition of the canonical momenta.

consistency algorithm outlined above must then be iterated until new constraints (*tertiary*, and so on) or restrictions on the u's can no longer be generated (or the theory is inconsistent).

At each step of the analysis we restrict more and more the physically accessible region of the phase-space and so the algorithm generates the following sequence of embedded (sub)manifolds:

$$\mathcal{P} \xrightarrow{\text{step } 0} \Sigma_0 \xrightarrow{\text{step } 1} \Sigma_1 \longrightarrow \cdots \longrightarrow \Sigma_f \equiv \Sigma \,, \tag{2.39}$$

where the final constraint submanifold $\Sigma_f \equiv \Sigma$ is determined by all the constraints, i.e. assuming that at the end of the Dirac's algorithm we get M new constraints

$$\chi_m(q,p) \approx 0 \qquad , \qquad m = 1, \dots, M \tag{2.40}$$

and denoting the set of all constraints (primary, secondary, and so on) $\{\phi_1, \dots, \phi_{N-K+M}\} = \{\varphi_1, \dots, \varphi_{N-K}, \chi_1, \dots, \chi_M\}$ with a uniform notation $\phi_j \approx 0$, $j = 1, \dots, J = N - K + M$, we have

$$\Sigma = \Big\{ (q,p) \in \mathcal{P} \;\Big|\; \phi_j(q,p) \approx 0 \,,\; j = 1, \dots, J = N - K + M \Big\} \subset \cdots \subset \Sigma_0 \subset \mathcal{P} \,. \tag{2.41}$$

The consistency conditions between the constraints lead to restrictions on the Lagrange multipliers u. In fact, we have the following inhomogeneous linear system

$$\dot{\phi}_j \approx \{\phi_j, H_0\} + u^k \{\phi_j, \phi_k\} \approx 0 \tag{2.42}$$

of J equations for the $K \leq J$ unknowns u^k. Provided the system is compatible (otherwise the dynamics would be inconsistent), the solution is given by $u^k = U^k + V^k$, where U^k is a particular solution of the inhomogeneous system and V^k represents the general solution of the associated homogeneous system $V^k\{\phi_j, \phi_k\} \approx 0$. This is expressed as a linear combination of linearly independent solutions $V^k = v^a V_a^k$, $a = 1, \dots, A = J - r$, where r is the rank of the homogeneous system assumed to be constant all over the constraint hypersurface. Thus, the general solution of (2.42) reads as

$$u^k \approx U^k + v^a V_a^k \,, \tag{2.43}$$

which can be inserted into (2.33) yielding the total Hamiltonian

$$H_T = H' + v^a \phi_a \qquad \text{with} \qquad H' = H_0 + U^k \phi_k \quad , \quad \phi_a = V_a^k \phi_k \tag{2.44}$$

whose terms respectively include the contributions to u^k coming from the consistency conditions and those that instead remain arbitrary (the remaining A arbitrary functions v^a).

2.2.2 Gauge Ambiguity of Dynamics: Presymplectic Structure

Another classification of constraints, that is physically more important than the one in primary and secondary constraints, is that of *first* and *second class* constraints according to the following definition:

First and second class function

A phase space function $f \in \mathcal{F}(\mathcal{P})$ is said to be *first class* if it has (at least weakly) vanishing Poisson bracket with all constraints, i.e. $\{f, \phi_j\} \approx 0 \ \forall j$. Otherwise, the function is called *second class*.

All the ϕ_a above are primary first class constraints by their definition. H_T is first class by the consistency requirement of all constraints to be preserved in time, hence by linearity H' in Eq. (2.44) is first class as well. The Poisson bracket of two first class constraints is also first class and it is thus *strongly* equal to a linear combination of first class constraints, say

$$\{\phi_i, \phi_j\} = C_{ij}^k \phi_k \ . \tag{2.45}$$

This shows that first class constraints close an algebra. The latter is not necessarily a Lie algebra since the coefficients C_{ij}^k might a priori be phase space functions and not necessarily constants (structure functions rather then structure constants). This is for instance the case of canonical general relativity [15–17].
The importance of first class constraints lies in the fact that first class primary constraints can be identified with the generators of infinitesimal gauge transformations[11], i.e. they change the canonical variables q, p but do not change the physical state of the system as reflected by the ambiguity left in the final form of the dynamics encoded in the unknown v^a in H_T. To show this, let us consider a phase space function f and its variation Δf along the infinitesimal evolution generated by H_T in (2.44) from t to $t + \Delta t$ given by (neglecting $\mathcal{O}(\Delta t^2)$ terms)

$$\begin{aligned} f(\Delta t) = f_0 + \dot{f}\Delta t = f_0 + \{f, H_T\}\Delta t \approx \underbrace{f_0 + \{f, H'\}\Delta t}_{\text{unique}} + \underbrace{v^a\{f, \phi_a\}\Delta t}_{\text{arbitrary}} \ , \\ \Rightarrow \quad \text{difference in evolution} \quad \Delta f = \Delta t \Delta v^a \{f, \phi_a\} =: \epsilon^a \{f, \phi_a\} \equiv \delta_\epsilon f \ . \end{aligned} \tag{2.46}$$

The ambiguity is thus generated by the combinations $\epsilon^a \phi_a$ with coefficients ϵ^a being entirely arbitrary, and states related by such transformation correspond to the same physical state. Dirac conjectured that all first class constraints (not only primary ones) are generators of gauge transformations[12]. It is thus possible to define an *extended Hamiltonian* H_E given by H' plus an arbitrary combination of all first class constraints

$$H_E = H' + \lambda^a \gamma_a \ , \tag{2.47}$$

with the index a running over a complete set of first class constraints, collectively denoted by γ_a. Strictly speaking, only the total Hamiltonian H_T follows directly from the Lagrangian.

[11]Second class constraints deserve a separate discussion and require the introduction of a new mathematical object, known as the Dirac bracket. Since in these lectures we will be dealing only with first class constraints, we will not discuss this topic here and refer the interested reader to the references given above.

[12]The status of such a conjecture is still disputed. A proof exists under simplifying regularity conditions that are generically satisfied (see Sec. 3.3.2 of [10]). It is however possible to construct counterexamples, but these are pathological (see e.g. [10]). The conjecture holds true for all physically relevant systems that have been studied so far. Moreover, in the quantisation of constrained systems all first class constraints are treated on equal footing.

Indeed, as discussed for instance in Ch. 3 of [10], the claim "(all) first-class constraints generate gauge transformations" refers to the extended action

$$S_E[q^i, p_i, \lambda^a] = \int \mathrm{d}t \, \left(p_i \dot{q}^i - H_0 - \lambda^a \gamma_a\right) \,, \tag{2.48}$$

with the γ_a first-class constraints and λ^a the Lagrange multipliers enforcing them. Solutions to the EOMs are tuples $(q(t), p(t), \lambda(t))$, and symmetries act on all these dynamical variables. The extended action (2.48) is invariant under the infinitesimal local transformation $\delta_\epsilon f = \epsilon^a \{f, \gamma_a\}$, for any $f(q,p) \in \mathcal{F}(\mathcal{P})$, only if the Lagrange multipliers are let to transfrom as $\delta_\epsilon \lambda^a = \dot{\epsilon}^a + C^a_{bc}\lambda^b \epsilon^c$, where $\{\gamma_b, \gamma_c\} = C^a_{bc}\gamma_a$. Such a set of symmetries reduces to the symmetries of the non-extended canonical action

$$S_C[q^i, p_i, \bar{\lambda}^a] = \int \mathrm{d}t \, \left(p_i \dot{q}^i - H_0 - \bar{\lambda}^a \gamma_a\right) \,, \tag{2.49}$$

where now the $\bar{\lambda}$ only refer to the primary constraints, only after imposing the gauge condition $\lambda^k = 0$ for all non-primary γ^k. It is the symmetries of the canonical action, not of the extended action, that directly translate to symmetries of the original Lagrangian action. The residual gauge symmetries of this action are those that preserve the conditions $\lambda^k = 0$ for the non-primaries and are in general generated by a specific subset of combinations of the first-class constraints that some references refer to as the gauge generator(s). The extended Hamiltonian H_E introduces more arbitrary functions of time, but its definition is more natural from the canonical point of view, since it allows to treat all of the gauge generators on the same footing. The dynamics generated by the three Hamiltonian functions H', H_T and H_E are are the same up to gauge transformations and as such are physically equivalent.

To be more precise about the above gauge ambiguity of constrained dynamics, let us go back to the geometric description developed in the previous sections. As discussed in Sec. 2.1, the regularity condition of the Lagrangian is equivalent to the closed 2-forms ω_L on $T\mathcal{Q}$ and ω on $T^*\mathcal{Q}$ to be non dengenerate (hence symplectic). Therefore, the EOMs $\iota_{X_H}\omega = \mathrm{d}H$ admit as solution a unique vector field X_H and there is a one-to-one correspondence between phase space points and physical states of the system. In the case of non-regular Lagrangians, the 2-form is instead degenerate, that is it has a non-trivial kernel and is thus a pre-symplectic structure. In fact, let us first prove the following claim:

> There is a one-to-one correspondence between the element of $\ker \omega_0$ (recall ω_0 is s.t. $(FL)^*\omega_0 = \omega_L$) and first-class combinations of primary constraints.

Proof. Denoting by $K < N = \dim \mathcal{Q}$ the rank of the Hessian matrix $\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$ and solving the first K equations of the system $p_i = \frac{\partial L}{\partial \dot{q}^i}(i = 1, \dots, N)$ for the velocities $\dot{q}^\alpha (\alpha = 1, \dots, K)$ in terms of $q^1, \dots, q^n, p_1, ..., p_K$ and the remaining unknown velocities $p_\rho, \rho = K+1, \dots, N$[13].

[13] We assume with no loss of generality that the first K rows and columns of the Hessian matrix identify a maximal nonsingular submatrix.

In other words, there are $N+K$ independent variables q^i, p_α on the submanifold $\Sigma_0 \subset T^*\mathcal{Q}$ which is defined by $N-K$ primary constraints of generic form

$$\varphi_\rho(q,p) = p_\rho - f_\rho(q^i, p_\alpha) \approx 0 \quad , \quad \rho = K+1, \ldots, N$$

which express the remaining dependent momenta $p_{K+1}, \ldots, p_N$ as functions of the independent q^i, p_α. The 2-form ω_0 on the primary constraint submanifold Σ_0 (where $\varphi_\rho \approx 0$) can be explicitly deduced from the 2-form ω on the full phase space $\mathcal{P} = T^*\mathcal{Q}$ as

$$\begin{aligned}
&\omega = \mathrm{d}p_\alpha \wedge \mathrm{d}q^\alpha + \mathrm{d}p_\rho \wedge \mathrm{d}q^\rho \ , \\
\Rightarrow \quad &\omega_0 = \imath^*\omega \qquad (\text{with } \Sigma_0 \overset{\imath}{\hookrightarrow} \mathcal{P} \text{ the embedding map of } \Sigma_0 \text{ into } \mathcal{P}) \\
&\quad = \mathrm{d}p_\alpha \wedge \mathrm{d}q^\alpha + \mathrm{d}f_\rho \wedge \mathrm{d}q^\rho = \mathrm{d}p_\alpha \wedge \mathrm{d}q^\alpha + \frac{\partial f_\rho}{\partial q^j}\mathrm{d}q^j \wedge \mathrm{d}q^\rho + \frac{\partial f_\rho}{\partial p_\alpha}\mathrm{d}p_\alpha \wedge \mathrm{d}q^\rho \ .
\end{aligned}$$

Given then a vector field $X \in \mathfrak{X}(\Sigma_0)$, say

$$X = X^j \frac{\partial}{\partial q^j} + X_\alpha \frac{\partial}{\partial p_\alpha}$$

we have

$$\imath_X \omega_0 = 0 \quad (X \in \ker \omega_0) \qquad \Rightarrow \qquad \begin{cases} X^\alpha = -X^\rho \frac{\partial f_\rho}{\partial p_\alpha} \\ X_\alpha = -X^\rho \frac{\partial f_\rho}{\partial q^\alpha} \\ X_\alpha \frac{\partial f_\rho}{\partial p_\alpha} + X^\rho \left(\frac{\partial f_\rho}{\partial q_\sigma} - \frac{\partial f_\sigma}{\partial q^\rho} \right) \end{cases}$$

from which, substituting the fist two equations into the third, we find the condition

$$0 = (\{f_\rho, f_\sigma\} + \{f_\sigma, p_\rho\} - \{f_\rho, p_\sigma\}) X^\sigma = \{\varphi_\rho, \varphi_\sigma\} X^\sigma \qquad (\text{on } \Sigma_0) \ .$$

where in the last equality we used the linearity of the PB. This shows that the elements of $\ker \omega_0$ are uniquely associated with first-class combinations $X^\sigma \varphi_\sigma$ of primary constraints. □

Going then through the Dirac procedure discussed in the previous section, we end up with the 2-form ω_f on the final constraint submanifold Σ_f ($\omega_f = \imath_f^*\omega$ with $\Sigma_f \overset{\imath_f}{\hookrightarrow} \mathcal{P}$) and the associated final form of the EOMs generated by H_T or H_E, say

$$\imath_{X_T}\omega = \mathrm{d}H_T \qquad \text{or} \qquad \imath_{X_E}\omega = \mathrm{d}H_E \tag{2.50}$$

would not identify just one solution but a family of solutions which differ from each other by a vector field that is in the kernel of ω_f[14]. This means that if we now start with any initial condition in Σ_f , then we can evolve along the flow of any of these vector fields. The evolution is thus not *deterministic* as we can find a solution starting at a given initial condition with any of the vector fields solving Eqs. (2.50), i.e., up to a vector field in the kernel. Therefore, all these vector fields represent the same evolution and we can put them

[14]This is the same thing we discussed in less geometric terms about the differential equation (2.42), the solutions of the complete equation differing by a solution of the homegeneous one

into an equivalence class $[X_f]$. All such vector fields correspond to different gauges and are all equivalent form the point of view of the Physics.

Thus, in the case of constrained system, there is no one-to-one correspondence between points on the constraint surface and physical states because physical states have to be interpreted as equivalence classes of points on the constraint surface. To see this, let us first recall that the stability requirement $\dot{\gamma}_a \approx 0$ amounts to say that the Hamiltonian vector fields X_{γ_a} associated to first-class constraints ($\iota_{X_{\gamma_a}}\omega = \mathrm{d}\gamma_a \Rightarrow \iota_{X_{\gamma_a}}\omega_f = 0$) generate curves which remain on the constraint surface. These curves are called *gauge orbits* and the vector fields X_{γ_a} are tangent to them. Moreover, since first class constraints (weakly) Poisson-commute not only with the Hamiltonian but also among themselves, the vector fields X_{γ_a} form an involutive distribution (the X_γ's are closed under Lie bracket[15]) so that, by Frobenius theorem, these vector fields generate at each point of the constraint submanifold a hypersurface called a *gauge leaf* whose dimension is equal to the number of first-class constraints. In other words, the constraints surface Σ_f embedded in $\mathcal{P} = T^*\mathcal{Q}$ is foliated by the gauge leaves as schematically depicted in Fig. 2.

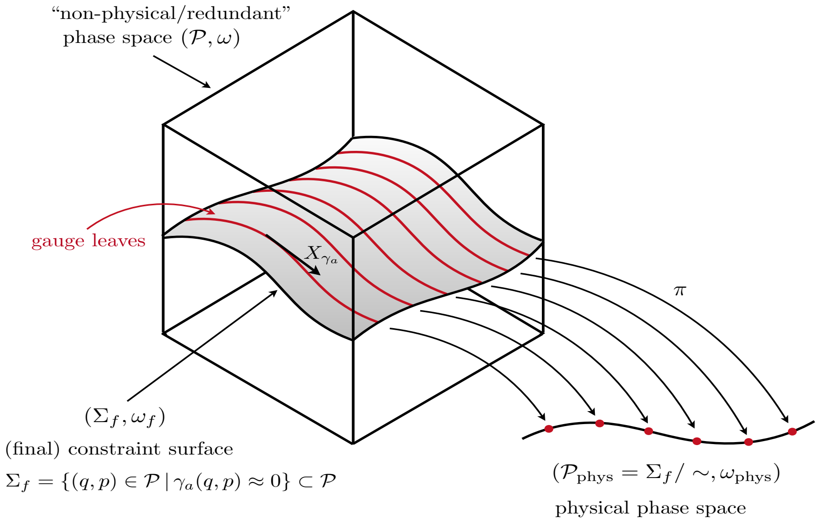

Figure 2. Constraint submanifold $\Sigma_f \subset \mathcal{P}$ foliated by gauge leaves (red) spanned at each point by the Hamiltonian vector fields X_{γ_a} associated with the first-class constraints γ_a (we assume there are no second-class constraints as e.g. is the case for Yang-Mills gauge theories or gravity). Points in phisycal phase space are identified with equivalence classes of points belonging to the same gauge leaf, that is points on the constraint surface related to each other by gauge transformations.

The constraints surface can be thus organised into equivalence classes by identifying all the points that lie on a gauge leaf, thus suggesting us the following physical interpretation:

[15] For any two vector fields $X, Y \in \ker\omega_f$ (i.e. $\iota_X\omega_f = \iota_Y\omega_f = 0$), $[X,Y]$ will also be in $\ker\omega_f$. Indeed, using Cartan's identity and the closure of ω_f, we have $\mathcal{L}_X\omega_f = \mathcal{L}_Y\omega_f = 0$, hence $0 = \mathcal{L}_Y(\iota_X\omega_f) = \iota_{[X,Y]}\omega_f$.

Physical phase space

There is a one-to-one correspondence between gauge leaves and physical states of the theory or equivalently, the physical phase space of the system $\mathcal{P}_{\text{phys}}$ is given by the space of such equivalence classes, i.e. by the quotient:

$$\mathcal{P}_{\text{phys}} = \Sigma_f / \sim \qquad \textit{physical phase space} = \frac{\textit{constraint surface}}{\textit{gauge transformations}} \tag{2.51}$$

The physical phase space $\mathcal{P}_{\text{phys}}$ comes to be equipped with a symplectic structure ω_{phys} defined by

$$\omega_{\text{phys}}(X,Y) := \omega_f(\tilde{X},\tilde{Y}) \quad X,Y \in \mathfrak{X}(\mathcal{P}_{\text{phys}}) \text{ s.t. } X = \pi_*\tilde{X}, Y = \pi_*\tilde{Y}, \tilde{X},\tilde{Y} \in \mathfrak{X}(\Sigma_f)\ , \tag{2.52}$$

where $\pi : \Sigma_f \to \mathcal{P}_{\text{phys}}$ is the map that sends each point in Σ_f to its $\mathcal{G}$-orbit.
Note that, starting from $\mathcal{P}$, pairs of canonically conjugate d.o.f. are removed by each first-class constraint: one via the algebraic restriction imposed by the equation $\gamma_a \approx 0$, and another one by the quotient w.r.t. gauge transformations. This is geometrically known as *symplectic reduction* and often denoted by a double quotient notation as $\mathcal{P}_{\text{phys}} = \mathcal{P}//\mathcal{G}$, with $\mathcal{G}$ the gauge group (cfr. Fig. 2).

Finally, since a physical observable O is a function on the phase space of the system which takes a definite value when the system is into a definite physical state, it follows that O has to take the same value at all the points on a given gauge leaf. In other words, only those phase space functions which Poisson-commute (at least weakly) with all first class constraints have a gauge-invariant physical meaning according to the following definition:

Dirac observables

A phase space function $O \in \mathcal{F}(\mathcal{P})$ is called a *Dirac observable* if it Poisson-commutes weakly with all first class constraints, i.e.

$$\mathcal{L}_{X_{\gamma_a}} O = \{O, \gamma_a\} \approx 0 \quad \forall\, a\ . \tag{2.53}$$

Dirac observables are thus the only phase space functions surviving after symplectic reduction and as such they parametrise the physical phase space of the system. In particular, recalling that at the end of the constraint analysis the undetermined multipliers in the Hamiltonian multiply first-class constraints (cfr. Eqs. (2.46), (2.47)), the condition (2.53) implies that the dynamical evolution of an observable does not depend on the arbitrary multipliers. In other words, unlike non-gauge invariant functions whose dynamics depend on the unknown multipliers, the theory is *deterministc* as long as we consider Dirac observables.

Summary: constrained systems and gauge symmetry

- singular Lagrangian, not all velocities can be solved for the momenta, Legendre transform leads onto primary constraints submanifold in phase space
- stability of constraints under evolution may lead to further constraints, consistency analysis progressively reduces the physically accessible region
- first-class constraints generate gauge transformations, the latters being associated with the flow of null directions of the presymplectic structure
- physical phase space identified with space of equivalence classes of points on the constraint surface lying on the same gauge leaf
- physical information encoded in gauge-invariant Dirac observables

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