Entanglement equilibrium and Semiclassical Einstein equations

Lecture # 14

In the lecture on BH thermodynamics, we had emphasised that the laws of BH mechanics (arising from purely classical GR considerations) can only be genuinely interpreted as laws of thermodynamics when we take into account quantum effects. This was evident from a finite Bekenstein–Hawking entropy and the existence of Hawking radiation and temperature. Such thermal features in BH spacetimes suggest also a fundamental quantum statistical origin for them. Then in the previous lecture, we essentially asked the question of whether thermality is a property more generally of spacetimes (i.e. gravity) rather than only of specific solutions, like stationary BHs and global horizons.

We saw that Jacobson's 1995 results addressed this question (in part). An important ingredient in his analysis was to work with a suitable notion of local neighbourhoods — local Rindler patches. We also saw a different notion of local nhds. in terms of causal diamonds, which has been found to be relevant in several studies especially in the context of deriving Einstein's eqns. (or higher curvature / modified theories of gravity) from more fundamental principles.

But our discussion was restricted to classical physics, and we recalled above that we cannot ignore quantum effects if we are to understand better the deep interplay b/w gravity and thermality.

In this lecture, we will focus on Jacobson's more recent [2016] results, based on the idea of entanglement equilibrium in local causal diamonds. The methods are mathematically straightforward, but the arguments are conceptually subtle — it is intriguing to see how a set of physically reasonable ingredients and certain (well defined, in the sense of allowing for concrete checks for verification, at least in principle) assumptions combine in just the right way to give the semiclassical Einstein's equations

— whether fully non-linear, or linearized, is still under debate. We'll briefly comment on this later.

In this lecture, we will carefully go through the [2016] analysis, and only briefly point out other relevant/follow-up studies in its context. The reason for focussing mostly on this analysis is because we think that (i) it is full of insights, without restricting too much to a specific approach or formalism, which could be of value elsewhere (ii) it could be hinting at aspects/features/ ingredients which could turn out to be important also for specific approaches to QG to make contact with known spacetime physics.

As suggested by some treatments of past lecture, in the context of causal diamond thermodynamics, this topic may not entirely be disconnected from the covariant phase space formalism. But here, we will restrict to the analysis as reported in [Jacobson 2016], also because to the best of our knowledge, an analogous semi-classical, approach independent analysis has not yet been carried out using covariant space methods.

Semi-classical treatment

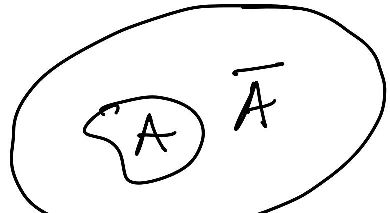
$\longrightarrow$ quantum matter, classical spacetime

To start with, let us look at an important property of quantum matter (regardless of any specific context of its application)

1^{st} law of entanglement entropy

Let $\psi \in \mathcal{H}$ arbitrary, full state

Consider a spatial region A, with complement $\bar{A}$



Reduced state on subsystem A : $\rho_A = \mathrm{Tr}_{\bar{A}} |\psi\rangle\langle\psi|$

Can write as $\rho_A = e^{-H_A}$

Modular Hamiltonian $H_A := -\ln \rho_A$

- in general, non-local and complicated
- in general, does not admit a geometric flow on spacetime

Quantifies Entanglement (quantum correlations) b/w A and $\bar{A}$: von Neumann entropy of reduced state

$\rightarrow$ Entanglement entropy $S_A = -\mathrm{Tr}(\rho_A \ln \rho_A)$

Infinitesimal variation : $\psi \mapsto \psi + \delta\psi$

$\uparrow$ 1st order variation

Aside: 1-parameter family of states $\psi(\varepsilon)$, with $\psi(\varepsilon=0) = \psi$, $\varepsilon \in$ small nhd. in $\mathbb{R}$ around 0.

Say $F = F[\psi]$. Then

$$\delta F = \left.\frac{\partial F(\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} .$$

(Standard method of functional variations)

$$\Rightarrow \delta S_A = -\mathrm{Tr}\left(\delta\rho_A \ln\rho_A\right) - \underbrace{\mathrm{Tr}\left(\rho_A\, \delta\ln\rho_A\right)}_{= \mathrm{Tr}(\delta\rho_A) = 0}$$

(fixed normalisation)

$$= \mathrm{Tr}(\delta\rho_A . H_A)$$

$$= \delta\, \mathrm{Tr}(\rho_A H_A)$$

$$= \delta\langle H_A\rangle \quad \text{(using } \delta H_A = 0\text{)}$$

$$\therefore \quad \delta S_A = \delta\langle H_A\rangle$$

"1st law of entanglement ent."

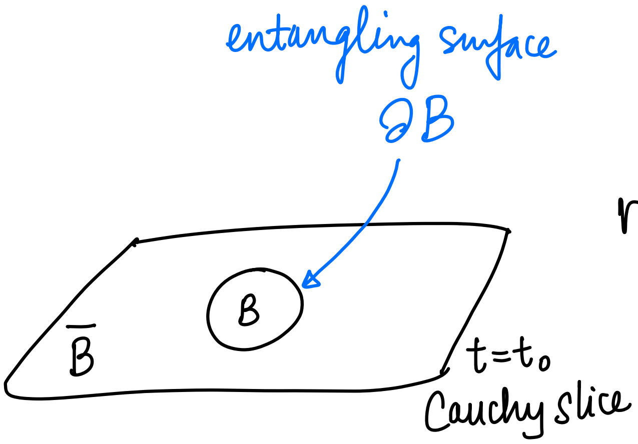
Relates first order variations in entanglement entropy and statistical avg. of modular Hamiltonian in the unperturbed state.

Now, consider: CFT in D-dim Minkowski, in vacuum state $|0\rangle\langle 0|$.

Ball-shaped spatial region, on some slice $t = t_0$

$B(\ell, \vec{x}_0)$

radius → ℓ; centre → $\vec{x}_0$ (flat spatial coords.)



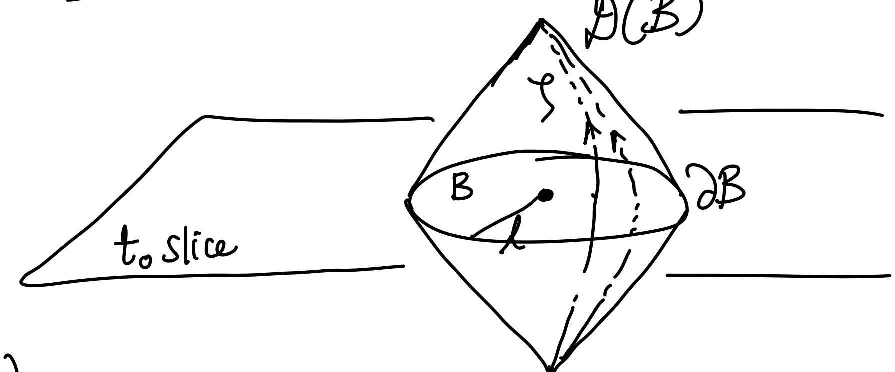
up to a constant, fixing the normalisation of ρ_B

$$\rho_B = \mathrm{Tr}_{\bar{B}} |0\rangle\langle 0| = e^{-H_B}$$

$$H_B = 2\pi \int_B (d^{D-1}x) \cdot \frac{\ell^2 - |\vec{x} - \vec{x}_0|^2}{2\ell} \cdot T_{00}(t_0, \vec{x})$$

$$= \int_B d\Sigma^\nu \, T_{\mu\nu} \, \xi^\nu$$

ξ^ν: generates conformal boost preserving $\mathcal{D}(B)$



ξ flow: null on $\partial\mathcal{D}(B)$

vanishing on ∂B, two vertex points

$$\xi = \frac{\pi}{\ell}\left[\left(\ell^2-(t-t_0)^2-|\vec{x}-\vec{x}_0|^2\right)\partial_t - 2(t-t_0)(x^i-x_0^i)\partial_i\right]$$

In fact, saw this in previous lecture

$$\xi = \frac{1}{2\ell}\left[\left(\ell^2-r^2-t^2\right)\partial_t - 2rt\,\partial_r\right]$$

for causal diamonds of geodesic balls

$\Longrightarrow$ 1st law of entanglement entropy in CFT:

$$\delta S_B = 2\pi\,\delta\int_B (d^{D-1}x)\,.\,\frac{\ell^2-\overset{\substack{\text{dist. from centre of } B\\ \downarrow}}{r^2}}{2\ell}\,.\,\langle T_{00}\rangle\,.$$

- for a ball-shaped spatial region B
- associated with CFT vacuum
- flat D-dim Mink.

Such contribution to $\delta S_{\substack{\text{entanglement}\\ \text{entropy}}}$ will be considered below.

In fact, at this point, the 1st law of entanglement entropy for CFTs

$$\delta S_B = \delta \langle H_B \rangle$$

can be used with AdS/CFT dictionary i.e. rewrite LHS and RHS above in terms of bulk geometric quantities, to make contact with gravitational eqns. in the bulk

→ Linearized (around AdS) Einstein's eqns.
[Lashkari, McDermott, Raamsdonk 1308.3716]

→ Linearized (around AdS) for higher curvature theories by considering Wald entropy (instead of only the leading order Bekenstein-Hawking formula)
[Faulkner, Guica, Hartman, Myers, Raamsdonk 1312.7856]

→ Extension to 2nd order, of Einstein eqns., (around AdS), without using duality
[Faulkner, Hael, Hijano, ... 1705.03026]

But here: focus on [Jacobson 2016] approach, without using AdS/CFT and not restricting to conformal matter.

Entanglement equilibrium

In the spirit of Bekenstein's Generalised Second Law that we saw before, the idea is to define a notion of equilibrium by stationarity of total Entanglement Entropy:

entropy maximization

$$\delta S_{tot} = 0 \qquad \text{"equilibrium"}$$

What is this variation with respect to?

[Jacobson 2016]: first order variation around local vacuum

Idea: notion of "vacuum as an eqm. state"

$$\delta S_{tot}\Big|_{\text{local vacuum}} = 0$$

$$\quad = \delta S_{UV} + \delta S_{IR}$$

assumption: clean separation of UV and IR d.o.f and their contributions to entanglement entropy

δS_{UV} : Contribution from geometry, keeping state fixed

in principle, this variation contribution to the total entanglement entropy includes also "UV" quantum gravity physics.

But here: semi-classical setup, thus consider only leading order

δS_{IR} : Contribution from quantum matter, keeping geom. fixed

→ use 1st law of entanglement entropy to include contribution from modular Hamiltonian

Entanglement eqm. and Semiclassical Einstein eqns.

$$\delta S_{UV} + \delta S_{IR}\Big|_{\text{local vac}} = 0 \quad \Longleftrightarrow \quad G_{ab} + \Lambda g_{ab} = 8\pi G\, \delta \langle T_{ab} \rangle$$

Will see that, cosmological const. term comes from using → Maximally Symmetric Spacetime (MSS)

variation wrt.

Local vacuum :

In $\underbrace{\text{small enough neighbourhoods}}$, around any $p \in M$

D-dim spacetime

* Specifically, use causal diamonds of geodesic balls
* introduce scale heirarchy, needed for conceptual consistency of arguments made

- spacetime looks like maximally symmetric
- quantum field like vacuum

This is [Jacobson 2016]'s proposal of a "good" choice of a local vacuum state of geometry and matter.
And further that the notions of local vacuum and local (entanglement, thus also quantum statistical) equilibrium coincide, at least at leading order.

i.e. in small enough neighbourhoods,

take any state $(g, |\psi\rangle)$ as a

small perturbation of $\underbrace{\text{local vacuum}}_{\text{where}}$

geom: MSS

matter: thermal Gibbs

These ideas are summed up in :

<u>Maximal Vacuum Entanglement Hypothesis</u> (MVEH)

"entanglement entropy in small geodesic balls is maximized at fixed volume in a locally maximally symmetric vacuum state of geometry and quantum fields"

in fact, we briefly came across this aspect of a fixed volume (of the geodesic ball B) in the previous lecture, for the 1st law of causal diamonds.
That such a condition is required was first stated in [Jacobson 2016] (for consistency, as we will see below) and later clarified further in treatments discussed in last lecture.

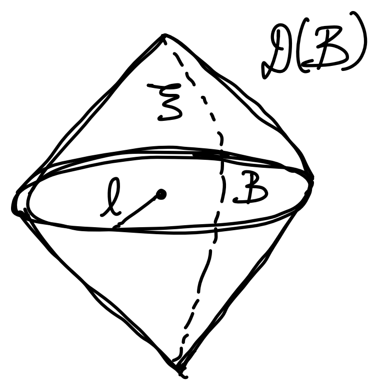
Setup: "small neighbourhood"

Spacelike geodesic ball $\mathcal{B}$, radius ℓ, centred at some $p \in M$

Select any timelike vector $u^a \in T_pM$ @ p, and shoot out geodesics of proper length ℓ from p in $(D-1)$-space ortgonal to u^a.

Scale heirarchy: $\ell \ll L_{curvature}, L_{QFT}, L_{excitations}$

Small enough, but not too small $\quad \ell \gg L_{Pl}$



Vary $\delta g, \delta\psi$ in $\mathcal{D}(\mathcal{B})$

$$\delta S_{tot} = \delta S_{UV} + \delta S_{IR} = 0$$

MVEH

Area deficit

Can encode the content of Einstein's eqns

$$G_{ab} = 8\pi G T_{ab}$$

in a statement about (a deficit in) the surface area of any small spacelike ball in presence of (positive) energy density in the ball.

Consider small geodesic ball B, radius $l \ll L_{\text{curvature}}$ centred @ $p \in M$

In Reimann Normal Coords. (RNC) around p:
(in lowest non-trivial order in $l/L_{\text{curvature}}$)

deficit

area of unit $(D-2)$-sphere

$$\delta A\Big|_V = -\frac{\Omega_{D-2}\, l^D}{2(D^2-1)} R$$

variation around flat (∵ using RNC)

area of ∂B

at fixed volume of Σ

spatial Ricci scalar @ p

Notice: @ fixed l, instead of V

$$\delta A\Big|_l = \left(\frac{D+1}{3}\right) \delta A\Big|_V$$

i.e. larger than $\delta A|_V$ by this factor

(It will be important to use $\delta A|_V$ and not $\delta A|_\ell$, for consistency — see below)

Now: $R = R_{ij}{}^{ij} = R - 2R_0{}^0 = 2G_{00}$

$$\Rightarrow \delta A|_V = \frac{-\Omega_{D-2}\,\ell^D}{(D^2-1)} G_{00} = -\frac{8\pi G\,\Omega_{D-2}\,\ell^D}{(D^2-1)} T_{00}$$

Einstein eqns.

If holds for all $p \in M$ and all timelike unit vectors @ p (i.e. all geodesic balls), then this encodes the full content of Einstein's eqns.

In fact, this also suggests that the associated thermodynamic temperature must be negative, since it's an area deficit for positive energy: heuristically,

$$\delta A \propto -\delta E$$

$$\rightarrow \delta S \propto -\delta E$$

$$\rightarrow \frac{\delta E}{\delta S} < 0$$

as energy increases, entropy decrease $\longleftrightarrow$ negative T

Now back to deriving EEqn. from MVEH :

Let us throw out EEqns. in their standard form ($\because$ we want to derive them), **but** keep its content through area deficits of $\mathcal{B}$.

And, recall postulate : local **MSS**

$\longrightarrow$ variation δ wrt MSS

Einstein tensor in MSS : $G_{ab}^{MSS} = -\lambda g_{ab}$

constant curvature scalar $\lambda = \left(\frac{D-2}{2D}\right)R$

using $R_{abcd} = \frac{R}{D(D-1)}(g_{ac}g_{bd} - g_{ad}g_{bc})$

Riemann curvature for any MSS

Now, variation around MSS

$$\Longrightarrow \delta A\Big|_{V,\lambda} = -\frac{\Omega_{D-2}\, l^D}{D^2-1}\cdot\left(G_{00} - G_{00}^{MSS}\right)$$

before, had $G_{00} - G_{00}^{flat} = G_{00}$

$$= -\frac{\Omega_{D-2}\, l^D}{D^2-1}\left(G_{00} + \lambda g_{00}\right)$$

Entanglement entropy variations in $\mathcal{D}(B)$

$$\delta g \longrightarrow \delta S_{UV} = \eta\, \delta A\big|_{V,\lambda} \quad \text{with } \eta \text{ constant}$$
$(\delta\psi = 0)$

Assume this form, with the below motivations

- Semi-classical treatment, without access to QG details
  → η finite, constant
- Take leading order Bekenstein-Hawking form above since
  → area scaling natural in any theory with a large density of states @ short distances
  → all other variations are to leading order also
- $\delta A\big|_{V,\lambda}$ @ fixed V, needed for consistency of derived relation
  $\eta = \frac{1}{4\hbar G}$ b/w η and G and
  for $\lambda \neq 0$ to get cosmological const. (see below)

$\delta\psi \longrightarrow \delta S_{IR} = \frac{2\pi}{\hbar} \delta \langle K \rangle$

$(\delta g = 0)$

1st law of entanglement entropy

modular Hamiltonian

i.e. local matter vacuum is $\rho = e^{-\beta K}$

where $\beta = \frac{2\pi}{\hbar}$

For CFT:

normal to B

$$K = H_\xi = \int_B d\Sigma_\mu \, T^{\mu\nu} \xi_\nu \quad \text{in } D\text{-dim Mink.}$$

$t=0$ plane $\rightarrow B$

$$\text{with } \xi = \frac{1}{2\ell}\left[(\ell^2 - r^2 - t^2)\partial_t - 2rt\partial_r \right]$$

Small $B \longrightarrow$ take $\langle T_{ab} \rangle$ const.

$$\Rightarrow \langle H_\xi \rangle = \langle T_{00} \rangle \int_B d\Sigma_0 \frac{1}{2\ell}(\ell^2 - r^2)$$

$$= \langle T_{00} \rangle \cdot \frac{\Omega_{D-2}\, \ell^D}{(D^2-1)}$$

But we are working with local MSS, not necessarily flat $\rightarrow$ problem? [1601.00528]

No, as long as $\ell \ll L_{\text{curvature}}$

Can check: for e.g. static dS patch

$$ds^2 = -\left(1-\frac{r^2}{L^2}\right)dt^2 + \frac{dr^2}{\left(1-\frac{r^2}{L^2}\right)} + r^2 d\Omega^2_{D-2}$$

Then, for B @ origin in above coords:

$$H_{dS} = 2\pi \int_{r\le \ell} d\Omega_{D-1}\, dr\, r^{D-2} \frac{(\ell^2-r^2)}{\ell(1-r^2/L^2)^3} T_{00}$$

$$\simeq 2\pi \frac{\Omega_{D-2}\,\ell^D}{D^2-1}\left(T_{00} + \underbrace{\left(\frac{2D-1}{D+3}\right)\frac{\ell^2}{L^2} T_{00}+\dots}_{\text{suppressed for } \ell \ll L}\right)$$

Thus, we are okay with taking H_ξ for CFTs ✓

For non-CFT: suppose $\exists$ UV fixed point
$\longrightarrow$ asymptotically CFT

Conjecture
for expectation values

$$\delta\langle K\rangle = \frac{\Omega_{D-2}\,\ell^D}{D^2-1}\left(\delta\langle T_{00}\rangle + \delta X\right)$$

spacetime scalar

to lowest order in ℓ.

Thus, take

$$\delta S_{IR} = \frac{2\pi}{\hbar} \cdot \frac{\Omega_{D-2}\, l^D}{D^2-1} \left(\delta\langle T_{00}\rangle + \delta X \right)$$

Semi-classical Einstein's eqn.

$$\delta S_{tot} = \eta\, \delta A \Big|_{V,\lambda} + \frac{2\pi}{\hbar}\, \delta\langle K\rangle$$

$$= \frac{\Omega_{D-2}\, l^D}{D^2-1} \left[-\eta \left(G_{00} + \lambda g_{00} \right) + \frac{2\pi}{\hbar} \left(\delta\langle T_{00}\rangle + \delta X \right) \right]$$

Then, imposing MVEH for all $p \in M$, timelike $u^a \in T_pM$

$\hookrightarrow$ recover all tensor components.

gives

$$\boxed{G_{ab} + \lambda g_{ab} = \frac{2\pi}{\hbar\eta} \left[\delta\langle T_{ab}\rangle + \delta X\, g_{ab} \right]} \qquad (*)$$

Apply ∇_a on both sides:

$\nabla_a G_{ab} = 0$ (Bianchi)

$\nabla_a (\lambda g_{ab}) = (\nabla_a \lambda) g_{ab}$ (metric compatible connection)

$\nabla_a \delta\langle T_{ab}\rangle = 0$ (conservation)

$$\nabla_a \left(\delta X\, g_{ab}\right) = \left(\nabla_a \delta X\right) g_{ab}$$

$$\Rightarrow (\nabla_a \lambda) g_{ab} = -\frac{2\pi}{\hbar\eta} (\nabla_a \delta X) g_{ab}$$

$$\Rightarrow \lambda = \frac{2\pi}{\hbar\eta} \delta X + \Lambda$$

Λ ← arbitrary integration (spacetime) constant

Plugging into (✱):

$$\boxed{G_{ab} + \Lambda g_{ab} = \frac{2\pi}{\hbar\eta} \delta\langle T_{ab}\rangle} \qquad (✱✱)$$

Consistency with Einstein's equs. $\Rightarrow \dfrac{2\pi}{\hbar\eta} = 8\pi G$

- Eqn (**) is independent of l ✓
  → causal diamond size an auxilliary parameter, and drops out, as it should

- $G = \dfrac{1}{4\hbar\eta}$ → derived (from consistency requirement), not assumed
  → stronger area density of entanglement entropy i.e. bigger η, weaker the gravity coupling strength i.e. smaller G

  "stronger vacuum entanglement implies weaker gravity, i.e. greater spacetime rigidity"

- If would have used $\delta A|_l$ (instead of $\delta A|_V$)

  then $G = \left(\dfrac{D+1}{3}\right)\dfrac{1}{4\hbar\eta}$ ← inconsistent with GR

  → follow-ups: causal diamond thermodynamics (saw in previous lecture)

  providing some more insight into the fixed volume condition which arose naturally there starting from Noether charge [2016-PRL-appendix, 1812.01596, ...]

$$\delta H_\xi = \int_{\partial B} \delta Q_\xi$$

$$\delta H_\xi^{(g)} + \delta H_\xi^{(m)} \qquad \frac{-\kappa}{8\pi G}\,\delta A$$

$(\because \xi$ conformal KVF$)$
so $\delta H_\xi \neq 0$

volume(B)

$\delta H_\xi^{(g)} \propto -\delta V$ for GR

$$\Rightarrow \delta H_\xi^{(m)} \propto -\left(\delta A - \frac{(D-2)}{l}\,\delta V\right)$$

$$= -\delta A\big|_V$$

"1st law of causal diamonds"

- Consider $(*)$: Four possible cases

(i) $X=0$, $\lambda=0$: CFT, variation wrt. flat

$\Rightarrow$ EEqns. w/o Λ

(ii) $X \neq 0$, $\lambda = 0$: non-CFT, δ_{flat}

$\Rightarrow$ EEqns. w/o Λ

and $\nabla_a \delta X = 0$ — too strong constraint
e.g. $X \propto T = T^\mu{}_\mu$

(iii) $X = 0$, $\lambda \neq 0$: CFT, δ_{MSS}

$\Rightarrow$ EEqns. with $\Lambda = \lambda = \left(\frac{D-2}{2d}\right) R$ (spacetime constant)

(iv) $X \neq 0$, $\lambda \neq 0$: non-CFT, δ_{MSS}

Saw case above already

- Conjecture involving X, for generic matter [Jacobson 2016]: this concerns only standard QFT $\Rightarrow$ is either true or false.

Follow-ups: explored using AdS/CFT [Casini, Galante, Myers 1601.00528, Speranza 1602.01380]

Causal diamond thermodynamics (last lecture) $H_\xi^{(m)}$ contains generic matter contribution, including Λ.

- Debate still on as to whether (**) describes full non-linear EEqns., or 1st order variations.

LHS looks like fully non-linear in metric but RHS includes 1st order variation in $T_{\mu\nu}$
$\longrightarrow$ how to understand this?

- Here : analysis with infinitesimal 1^{st} order variations

→ considers finite but small variations to coherent states [Varadarajan 1602.0010] and identifies certain implicit assumptions in [Jacobson 2016] by considering 1^{st} and 2^{nd} order variations in S and $T_{\mu\nu}$.

Further reading

- Jacobson 2016 [arXiv:1505.04753]
- Some AdS/CFT + related literature:
  1312.7856, 1308.3716, 1705.03026
  1601.00528, 1602.01380
- Varadarajan 2016 [1602.00106]
- Jacobson, Visser 2018 [1812.01596]