13 Spacetime thermodynamics and Einstein's equations

As emphasised in the previous lecture, the fact that black hole spacetimes come equipped with a temperature and thermodynamic entropy suggests a deep interplay between gravity, quantum mechanics, and thermodynamics[6]. Shortly after the pioneering work by Bekenstein, Hawking, and many others which revealed that the analogy between the laws of black hole mechanics and thermodynamics was actually more than just an analogy (this was especially due to the discovery of Hawking radiation), it was realised that other solutions of Einstein's equations characterised by the existence of global horizons such as de Sitter and Rindler horizons also have thermodynamic properties. In particular, the fact that the latter are observer-dependent and therefore could be anywhere suggests that such thermodynamic features of gravity might actually be more generally applicable concepts in spacetime, i.e. black holes are not necessarily required and one is rather dealing more generally with a notion of *horizon thermodynamics*. In fact, given this situation, few questions naturally arise: how did classical general relativity know that spacetime geometric notions such as the horizon area and surface gravity turn out to be related to thermodynamic notions respectively as a form of entropy and temperature? Black hole, de Sitter, and Rindler horizons are global horizons, but how local can the notions of horizon thermodynamics be applied? What is special about horizons?

Taking this idea significantly further, in his 1995 paper entitled "*Thermodynamics of spacetime: The Einstein equation of state*" Ted Jacobson attributed thermodynamic properties even to local Rindler horizons, that is planar patches of certain null congruences passing through arbitrary spacetime points, which are thus not event horizons in any global sense. The locality of local Rindler "horizon" has the effect that local equations follow from thermodynamic equations. Specifically, Jacobson tried to address the above questions by turning the *geometry* $\rightarrow$ *thermodynamics* logic around and deriving the Einstein equations from the proportionality of entropy and horizon area together with Clausius fundamental relation $\Delta Q = T\Delta S$ connecting heat Q, entropy S, and temperature T. From such a new perspective, Einstein's field equation comes to be an equation of state, the latter arising in the thermodynamic limit as a relation between thermodynamic variables whose validity depends on the existence of local equilibrium conditions.

As Jacobson's derivation of the Einstein equation of state has been a profound source of inspiration for many research directions in both classical and quantum gravity, the main part of this lecture will be devoted to review such a remarkable result in its original formulation trying to put the emphasis on the key ideas and assumptions. We will then comment on the main lesson and conceptual implications that it provides, and give some outlook on the later extensions of the original result, one of which will be the subject of the following lectures. More specifically, in the final part of the lecture, we will discuss the extension of the first law to causal diamonds. Such an extension will be in turn needed in the following lectures to study a later improvement again by Jacobson 20 years after its original idea of deriving Einstein's equations from thermodynamic equilibrium, the latter being replaced by the so-called first law of entanglement equilibrium when both variation of the geometry and of the state of quantum matter fields are included, thus leading this time to semi-classical Einstein's equation.

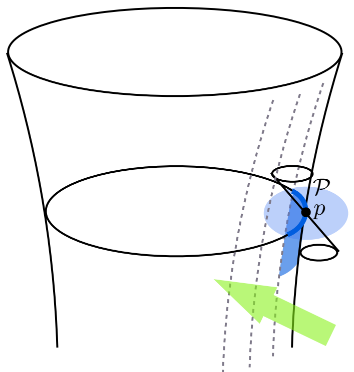
[6] We recall from Lecture 12 that the famous Bekenstein-Hawking area-entropy relation involves Boltzmann constants k_B, Planck constant $\hbar$, as well as Newton gravitational constant G.

13.1 Jacobson's original derivation of the Einstein equation of state

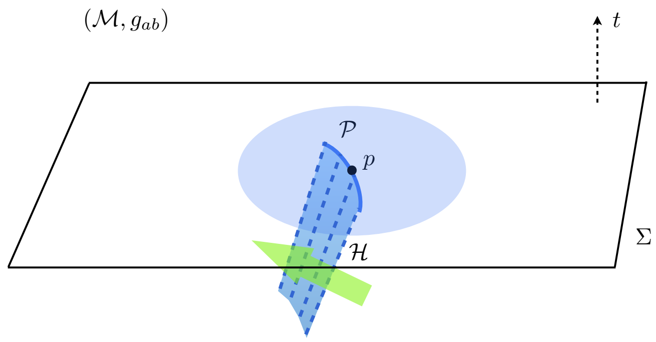
The first step in Jacobson's analysis is to characterise the system we are interested in and the relevant (thermodynamic) quantities.

13.1.1 Local causal horizons as thermodynamic systems

The basic idea and setup, which is schematically depicted in Fig. 13.1a and 13.1b below, can be understood as follows. If we have a black hole horizon, we know that BH horizons satisfy the laws of thermodynamics. In particular, as discussed in the previous lecture and pictorially represented in Fig. 13.1a, the horizon is ruled by null geodesics (it is a null hypersurface), at any given point of the horizon we have a light-cone, and when matter flows into a black hole the horizon area increases and it does so in a way which is consistent with the laws of thermodynamics. The puzzle is then: how does general relativity have the structure to make a BH behave thermodynamically? According to Jacobson, the answer is to be found into the interplay of two equivalence principles: the *spacetime equivalence principle*, according to which any spacetime looks flat in a sufficiently small neighborhood of a point, and the *QFT equivalence principle*, according to which all (matter) states are vacuum states in small neighborhoods. These suggest us to forget about the fact that we have a BH and just look at the neighborhood of a point p. The latter could be the neighborhood of a point anywhere in any spacetime, whether there is a black hole or not. The idea is then to analyse the process locally in a small neighborhood $\mathcal{P}$ of any spacetime point p using the above equivalence principles. With this logic in mind, we can then construct a similar setup anywhere in spacetime by looking at sections of local causal horizons of a point p and studying the change of the area of such a local horizon as energy flows through it (cfr. Fig. 13.1b). But what do we precisely mean by local horizons and what kind of energy are we referring to?



(a) A point p on a cross-section of the black hole horizon with its light-cone. When matter flows into the black hole, the horizon area increases and, consistently with the entropy-area relation, the horizon entropy increases.



(b) According to the equivalence principle, locally in any neighborhood of a point p – regardless of it being on a black hole horizon or any other (non-singular) point of whatever spacetime – the geometry is approximately flat. In a sufficiently small neighborhood $\mathcal{P}$ of any spacetime point p there exists a local Rindler horizon $\mathcal{H}$ given by the null hypersurface identified by the past-pointing normal null congruences emanating from $\mathcal{P}$.

Let us then try to identify more explicitly the gravitational system we would like to study. Let p be an arbitrary point in a D-dimensional spacetime $\mathcal{M}$ with arbitrary metric g_{ab}. Let then p be located on a space-like codimension-1 hypersurface Σ w.r.t. some foliation of spacetime and let us consider a nearly flat space-like region $\mathcal{P}$ containing the point p. By nearly flat we mean that the region $\mathcal{P}$ is assumed to be a small enough neighborhood of the spacetime point p so that, according to the equivalence principle, it can be viewed as a piece of flat spacetime. More precisely, this amounts to say that the null congruences emanating from and normal to $\mathcal{P}$ have initial vanishing expansion θ and shear σ_{ab} at p to first order in the distance from p obtained by a leading order expansion of the metric in Riemann normal coordinates (i.e. $g_{ab} \approx \eta_{ab} + \frac{1}{3}R_{abcd}(p)x^b x^d + \dots$). Such an assumption is crucial to consider the system of interest to be in local equilibrium and we will come back to this point later. Before going to this, what do we mean by "the system"? To this aim, let us consider the past directed null congruence normal to $\mathcal{P}$. Denoting by λ the affine parameter along the congruence, $k^a = \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^a$ is the tangent vector to the null congruence and we set $\lambda = 0$ at p. The points of such a null congruence generate a null surface $\mathcal{H}$ called the (past-directed) *lightsheet* emanating from $\mathcal{P}$. The cross-sectional area $\mathcal{A}$ of the lightsheet is related to the expansion θ of the null congruence via the relation $\theta = \frac{1}{\mathcal{A}}\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}\lambda}$.

So far the whole discussion only relies on geometric notions, so how do we translate these into thermodynamic concepts? The key observation is that such a lightsheet plays the role of a "local Rindler horizon" of $\mathcal{P}$ as it serves as a local causal horizon for an uniformly accelerated observer hovering just inside it (cfr. red hyperbola in Fig. 13.2). There are local Rindler horizons in all null directions through any spacetime point and we think of such local causal horizons (which are not global event horizons) as defining our system of interest, namely the part of spacetime beyond the Rindler horizon. The idea is then to associate a *local gravitational thermodynamics* with such causal horizons. Thermodynamic properties of spacetime come then to be tied together with the causal structure and, in particular, with the existence of local causal barriers associated with the (null) boundary of the past of a small neighborhood of any non-singular point in spacetime. In Jacobson's own words: *"...the "system" being the degrees of freedom beyond the horizon [...] The outside world is separated from the system not by a diathermic wall, but by a causality barrier."*

The relevant thermodynamic properties of such a system are then going to be heat, entropy, and temperature as measured by the above local Rindler observers. Indeed, matter entering or leaving the system, as measured w.r.t. the locally accelerating observers, is inntepreted as *heat Q*. The reason for such an interpretation in terms of heat flow rather than some other energy flux relies on the standard notion of heat from ordinary matter thermodynamics as measuring the flow of energy into macroscopically unobservable degrees of freedom[7]. Similarly, in the gravitational context of spacetime dynamics, *heat* is defined as the energy that flows across a causal horizon. It can be felt via the gravitational field it generates (the lightsheet cross-sectional area is subject to a change as matter flows through the lightsheet), but its explicit form is unobservable from outside the horizon. Moreover, building up on Bekenstein's work on black holes which as it was already stressed in the previous lecture was heavily inspired by Jaynes' statistical mechanic viewpoint on entropy, Jacobson argues that it is possible to associate entropy with causal horizons in a similar spirit to which we associate a notion of entropy to any information barrier. Such barriers are causal barriers in the context of gravity,

[7] For instance, the fact that we can heat water up relies on the existence of some internal mechanism to store energy into water's microscopic deegrees of freedom, the specific details of the latter being unobservables on macroscopic scales.

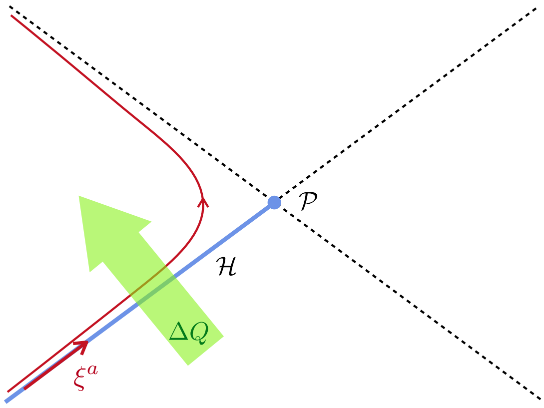


Figure 13.2: Spacetime diagram of the relevant quantitites involved in local horizon thermodynamics – Each point in the diagram represent a codimension-2 spacelike surface. The thermodynamic system of interest is identified with the part of spacetime beyond a local Rindler horizon $\mathcal{H}$ (blue null hypersurphace) of a codimension-2 surface $\mathcal{P}$. The (matter) energy flux across the causal horizon is interpreted as a heat flow (green arrow). The entropy of the system beyond the horizon is large due to the vacuum fluctuations of quantum fields and is proportional to the cross-sectional area of the horizon. The temperature assigned to the system is then identified with the local Unruh temperature associated with a (uniformly) accelerated observer inside the horizon whose worldline is represented by the red hyperbola in the diagram. The latter comes to be a local causal barrier for such an observer, the role played by the former motivating the assignment of a notion of entropy with the system. For consistency, the energy defining the heat flow across the horizon is measured by the same local Rindler observer.

the hidden information residing in the correlations between vacuum fluctuations just inside and outside of the horizon. Motivated by the property of vacuum QFT entanglement entropy being proportional to the horizon area (in units of an a priori unspecified fundamental cutoff length ℓ_c which is much smaller than the curvature radius of spacetime), Jacobson assumed then the entropy of any local causal horizon to be proportional to the cross-sectional area of the lightsheet. This is in fact also the only quantity in the above local geometric setup which can change under the physical process of matter flowing through the lightsheet and it is an extensive quantity for the horizon as entropy is expected to be[8]. This is the notion of entropy to be used in local horizon thermodynamics and, as we will see in the next subsection, consistency with thermodynamics requires the cutoff length ℓ_c to be of order of Planck length. Finally, it remains to identify the temperature of the system into which the heat is flowing. In this respect, since the (large) entropy assigned to the system is due to the vacuum fluctuations of quantum fields and, according to the Unruh effect, such vacuum fluctuations have a thermal character from the perspective of a uniformly accelerated observer inside the horizon, the temperature of the system can be taken to be the the Unruh temperature associated with such a local Rindler observer. For consistency, the energy defining the heat flow across the horizon is then measured by the same local Rindler observer as schematically represented in Fig. 13.2.

Note that:

[8] We may also think of the entropy-area relation of black hole thermodynamics as providing us with a further motivation for assuming the entropy of the local causal horizon to be given by its area. However, in its original paper, Jacobson is explicitly refraining himself from doing so as it would have been a circular argument in his main logic to use the derivation of Einstein's equations as equation of state as an argument to account for the occurrence of the thermodynamic-like behaviour of black holes.

- even though different accelerated observers will measure different results, the acceleration diverges as their worldlines approach the horizon, hence the associated temperature and energy flux also diverge but their ratio has a finite limit. Such a limit makes the analysis of the local horizon thermodynamic process described above as local as possible.
- the system defined by any causal horizon will in principle not be in equilibrium as the horizon is in general expanding, contracting, or shearing. As anticipated at the beginning of this section, here is where the assumption of the null congruence emanating from and normal to $\mathcal{P}$ to have initial vanishing expansion θ and shear σ_{ab} at p at leading order in the Riemann normal coordinate expansion comes into the game. This in fact amounts to say that the system beyond the local Rindler horizon of $\mathcal{P}$ is instantaneously stationary at p compatibly with the temperature assigned to it being the constant Unruh temperature proportional to the uniform acceleration of the local Rindler observers measuring it. Therefore, similarly to the case of global Rindler spacetime, the local Rindler horizon is a constant temperature system, and is thus in (local) thermal equilibrium.

13.1.2 Gravity from local horizon equilibrium thermodynamics

With such a thermodynamic rephrasing of local geometric notions in spacetime, the main step in Jacobson's argument is to (re)derive Einstein's equation from the fundamental Clausius relation applied to the system in local equilibrium described above.

> *"The equilibrium thermodynamic relation $\Delta Q = T\Delta S$, as interpreted here in terms of energy flux and area of local Rindler horizons, can be satisfied only if gravitational lensing by matter energy distorts the causal structure of spacetime in just such a way that the Einstein equation holds."*

In their most essential form, in fact, Einstein's equations express the relation between the matter energy content of spacetime and the geometry of spacetime itself dynamically influencing each other. The assumption of the entropy being given by the cross-sectional area of the local causal horizon, on the one hand, brings in the geometry. The identification of the energy flux across the causal horizon as a kind of heat flow, on the other hand, brings in the matter energy-momentum tensor. The horizon area changes as matter flows through it so that the local equilibrium condition expressed by the Clausius equation (thermo)dynamically relates geometry and matter. Remarkably, this local equilibrium process around a spacetime point p does so in such a way that Einstein's equations are satisfied at p. As the whole argument holds locally at any point in spacetime, the tensorial nature of the Einstein's equation is then recovered.

Since in a small neighborhood of any point the spacetime locally appears flat, due to the Riemann normal coordinate expansion, we retain local Poincaré isometries of flat space. These include the Lorentz boosts which can be seen as Rindler time translations for a locally accelerating observer. The local Rindler observer will have a local Rindler horizon (the lightsheet $\mathcal{H}$). Since the Killing vector is only Killing within $\mathcal{O}(x^2)$ of the Riemann normal coordinate expansion, there is an approximate local Killing vector field generating boosts orthogonal to $\mathcal{P}$ and vanishing at $\mathcal{P}$ given by

$$\xi^a = -\lambda\kappa k^a \ , \tag{13.1}$$

with κ the uniform acceleration of the Killing orbit on which the norm of ξ^a is unity, k^a the tangent vector to the horizon generators, and λ the affine parameter of the null congruence spanning the lightsheet identified with the local horizon. In our convention, λ is chosen to vanish at $\mathcal{P}$ and negative to the past of $\mathcal{P}$ so that ξ^a is future pointing to the inside past of $\mathcal{P}$ (cfr. Fig. 13.2). As anticipated above, the local Rindler observers will measure a constant temperature ($c = 1$)

$$T = \frac{\hbar\,\kappa}{2\pi}\;, \tag{13.2}$$

which, according to the Unruh effect, characterises the thermal nature w.r.t. the boost Hamiltonian of any QFT vacuum state at very short distances.

According to the discussion of the previous subsection, the heat flow across the local horizon measured w.r.t. the local Rindler observer is identified with the boost-energy flux that flows into the macroscopically unobservables (from the perspective of the local accelerating observer) degrees of freedom behind the lightsheet, namely

$$\Delta Q = \int_{\mathcal{H}} \mathrm{d}\Sigma^a\, T_{ab}\xi^b = -\kappa \int_{\mathcal{H}} \mathrm{d}\lambda\, \mathrm{d}\mathcal{A}\, \lambda\, T_{ab}k^a k^b\;, \tag{13.3}$$

where $T_{ab}\xi^b$ is the boost-energy current of the matter, with T_{ab} the matter energy-momentum tensor, and $\mathrm{d}\Sigma^a$ is the surface area element of the local Rindler horizon. In the second equality we used the expression (13.1) for the local approximate boost Killing field and the relation $\mathrm{d}\Sigma^a = k^a \mathrm{d}\lambda \mathrm{d}\mathcal{A}$, with $\mathrm{d}\mathcal{A}$ the area element of the codimension-2 cross-section of the horizon. Note that, in the thermodynamic limit in which Eq. (13.3) is assumed to be written, quantum fluctuations in the matter energy-momentum are neglected. Moreover, as the temperature (13.2) and the heat flow (13.3) have the same scaling behaviour under a constant rescaling of the boost Killing field ξ^a, there is no scale ambiguity when imposing Clausius relation.

The matter-energy flow across the horizon produces a change $\Delta\mathcal{A}$ in the cross-sectional area of the latter and in turn a change ΔS of entropy. The key assumption of the entropy to be proportional to the horizon area in fact leads us express the entropy variation of a piece of our local horizon as

$$\Delta S = \eta\, \Delta\mathcal{A}\;, \tag{13.4}$$

for some undetermined dimensional constant η. The last ingredient we need is then to evaluate the r.h.s. of such a change in the entropy of the system. The infinitesimal variation of the cross-sectional area of a geodesic congruence is related to the expansion as $\theta = \frac{1}{\mathcal{A}}\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}\lambda}$, we have

$$\Delta\mathcal{A} = \int_{\mathcal{H}} \mathrm{d}\lambda\, \mathrm{d}\mathcal{A}\, \theta\;. \tag{13.5}$$

Now, recalling from Lecture 10 the Raychaudhuri's equation governing the "evolution" of the expansion $\theta(\lambda)$ as a function of the affine parameter λ along the geodesic congruence

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = -\frac{1}{(D-2)}\theta^2 - \sigma^2 + \omega^2 - R_{ab}k^a k^b\;, \tag{13.6}$$

together with the fact that the twist $\omega_{ab} = 0$ for hypersurface orthogonal geodesic congruences generating our null hypersurface, and that our local Rindler horizon was chosen to be instantaneously stationary at $\mathcal{P}$ (θ and σ_{ab} vanish at leading order in the Riemann normal coordinate expansion around p), the θ^2 and (squared) shear $\sigma^2 = \sigma_{ab}\sigma^{ab}$ terms give higher order contributions compared to the the one coming from the last term so that near $\mathcal{P}$ equation (13.6) yields

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = -R_{ab}(p)k^a k^b \qquad \Rightarrow \qquad \theta = -\lambda R_{ab}(p)k^a k^b\;, \tag{13.7}$$

where the integration constant in the expression of θ has been set to zero compatibly with the requirement of θ to be vanishing on $\mathcal{P}$ ($\lambda = 0$) at leading order in the Riemann normal coordinate expansion.

Plugging then Eqs. (13.2),(13.3),(13.4), (13.5), and (13.7) into Clausius equation $\Delta Q = T\Delta S$, we get

$$\kappa \int_{\mathcal{H}} \mathrm{d}\lambda\, \mathrm{d}\mathcal{A}\, \lambda\, T_{ab}(p) k^a k^b = \eta \frac{\hbar\kappa}{2\pi} \int_{\mathcal{H}} \mathrm{d}\lambda\, \mathrm{d}\mathcal{A}\, \lambda\, R_{ab}(p) k^a k^b \,. \tag{13.8}$$

Invoking the freedom in choosing the null surface $\mathcal{H}$ and the tangent vector k^a to the null geodesic congruence generating it, we can equate the integrands to get

$$T_{ab}(p) k^a k^b = \eta \frac{\hbar\kappa}{2\pi} R_{ab}(p) k^a k^b \,, \tag{13.9}$$

for all k^a, or equivalently (recall that k^a is a null vector $g_{ab}k^a k^b = 0$)

$$R_{ab}(p) + f(p)\, g_{ab}(p) = \frac{2\pi}{\hbar\eta} T_{ab}(p) \,, \tag{13.10}$$

for some unknown function f. The latter can be further specified by using the conservation of energy-momentum tensor ($\nabla^a T_{ab} = 0$) and the contracted Bianchi identity ($2\nabla^a R_{ab} = \nabla_b R$), thus yielding $f = -\frac{1}{2}R + \Lambda$, for some integration constant Λ. Therefore, making the identification $\eta = \frac{1}{4\hbar G}$ on the r.h.s. of Eq. (13.10), this leads us to the Einstein's equation holding about the point p, namely

$$R_{ab}(p) - \frac{1}{2} R(p) g_{ab}(p) + \Lambda g_{ab}(p) = 8\pi G T_{ab}(p) \,, \tag{13.11}$$

and in turn, since the point p is completely arbitrary, the above argument holds about any other point (as long as no caustic points or singularities are involved) and Einstein's equations hold throughout the entire spacetime. Note that, consistently with the Bekenstein-Hawking formula, the relation $\eta = \frac{1}{4\hbar G}$ leads us to identify the proportionality constant between entropy and area as $\eta^{-1} = 4\ell_p^2$ with $\ell_p = (G\hbar)^{1/2}$ the Planck length (in units in which $c = 1$).

13.2 Discussion and remarks

The above analysis shows that local Einstein's equations are a geometric consequence of applying thermodynamic principles to local horizons in any spacetime. This suggests that, just like the hydrodynamic limit of water, which going back to Boltzmann's arguments can be heated up because of some mechanism to store energy into its atomic microscopic constituents and exhibits thermodynamic properties as collective macroscopic behaviour of the latter, classical spacetime dynamics arises from some more fundamental microscopic theory of spacetime. Such an emergent viewpoint has been on the one hand a source of inspiration for classical "cousin" approaches (as e.g. Erik Verlinde's entropic gravity or Padmanabhan's emergent gravity paradigm) trying to consolidate the idea that either the gravitational force in general or more specifically the gravitational field equations can be understood as an effective macroscopic description of statistical mechanic nature of some collective phenomenon in a system composed of several degrees of freedom, which is however independent of the microscopic details of the system[9]. On the other hand, Jacobson's result and its conceptual implications for the existence

[9] We refer the interested reader to the bibliography at the end of this lecture for reviews on the topic which contain the relevant references.

of a more fundamental microscopic texture of spacetime become one of the main motivations shared by several quantum gravity approaches which, despite of their conceptual and technical differences, seek for candidates of such underlying microscopic structure of spacetime. Jacobson himself at the end of his 1995 paper argues that the above derivation of Einstein's equations might provide some interesting insight for quantum gravity. Indeed, a key ingredient in the above derivation of Einstein's equations was the imposition of local equilibrium conditions. The validity of Einstein's equation is then tied together with the validity of local equilibrium conditions. Therefore, just like for propagation of sound waves for which sufficiently high wave frequencies would break equilibrium, one might expect that for sufficiently high disturbance of the gravitational field, the latter would no longer be described by the Einstein's equations. According to Jacobson, this is not because of some quantum gravitational nature of the metric itself becoming relevant at certain scales, but rather because local equilibrium conditions would break down at those scales. This suggests that, even though the *emergent* Einstein's equations of state are ultimately describing (the thermodynamic limit of) some quantum mechanical phenomenon, it might not be correct to canonically quantise Einstein's equations, the hope being instead to gain new insights by exploring and trying to understand the *non-equilibrium nature/sector* of spacetime physics.

However, remaining in the realm of classical physics which is the main focus of this course, the above arguments seem to suggest that some genuine statistical mechanical properties of gravity might be the fundamental reason why black holes, global, and local horizons exhibit thermodynamic-like behaviours. We might then wonder how general Jacobson's argument is. This question boils down to address the following points: 1) does it apply also to other settings and generic theories of gravity rather than just (global or local) horizons and Einstein's general relativity? 2) What are the key assumptions? As for the first point, the answer is yes. Changing the entropy functional in the above argument leads to other gravitational field equations. For instance, the inclusion of dissipative contributions to the entropy (curvature corrections) leads to $f(R)$ modified theories of gravity. Replacing the simple Bekenstein-Hawking entropy-area relation with Wald's entropy leads to the field equations of generic diffeomorphism-invariant theories of gravity (including higher derivatives theories). People have also considered other kind of settings as e.g. future stretched lightcones and causal diamonds. We will of course have no time to cover all this generalisations in detail here so that we refer the reader to the further readings at the end of this lecture where references for each of these topics are provided. Among the various extensions originating from Jacobson's original result, in the remaining part of this lecture we will focus on the one which is relevant for the purposes of this course. That is, causal diamonds thermodynamics using Wald's entropy. The reason for such a choice is twofold. First of all, this setting will turn out to be useful in the next lecture and, second, it has some connection with the covariant phase space formalism on which the entire class has been focused. Moreover, this shares some of the key assumptions of Jacobson's original argument, namely the entropy-area relation and the use of Clausius' relation. The first assumption can be considered to be a milder assumption as motivated by black hole entropy, vacuum QFT entropy in presence of a UV cutoff, and Wald entropy. The second assumption based on Clausius' equality is more delicate and entailed to the validity of the local equilibrium approximation. In fact, in standard thermodynamics, Clausius' relation in generically given by the inequality $\Delta Q \leq T\Delta S$ and both reversible and irreversible contributions enter the entropy variation. The equality is obtained only when, in the thermodynamic process under consideration, there is no irreversible change in the entropy of the system. Clausius' equality then only applies to variations between nearby states of local thermodynamic equilibrium. The heat exchange process interpretation for the matter energy-flux through $\mathcal{H}$ relies on the latter being happening slowly enough such that the exchange process (and hence the entropy variation) can be thought of as being totally reversible[10].

13.3 Gravity from the first law of causal diamonds

As anticipated, we will now move our discussion to causal diamonds (CDs). These have spherical subregions, admits conformal Killing vector fields (CKVFs) whose flow preserves the diamond, and their boundary defines a conformal Killing horizon with a well-defined and constant notion of surface gravity[11]. Constant surface gravity allows us to interpret the causal diamond as a system in thermal equilibrium. The goal then is to assign thermodynamic quantities such as temperature and entropy to the causal diamond and, assuming proportionality between entropy and area, derive the gravitational field equation for a generic classical theory of gravity from Clausius' relation.

13.3.1 Geometric preliminaries: Causal diamonds and their conformal isometries

At any point p in D-dimensional (flat) spacetime, choose an arbitrary time-like unit vector U^a and consider the $(D-1)$-dimensional spacelike geodesic ball $\mathcal{B}$ generated by shooting out geodesics of length ℓ out of the point p in all orthogonal directions to U^a (cfr. Fig. 13.3 below). A causal diamond (CD) associated with the geodesic ball $\mathcal{B}$ around the spacetime point p is defined as the union $\mathcal{D}^+(\mathcal{B}) \cup \mathcal{D}^-(\mathcal{B})$ of the past and future domains of dependence of the spatial ball $\mathcal{B}$ and the codimension-2 boundary $\partial\mathcal{B}$ of $\mathcal{B}$ is usually referred to as the *edge* of the diamond (see Fig. 13.3). For CDs in generic spacetimes, one considers a system of Riemann normal coordinates (RNC) based at p and, assuming the radius ℓ of the geodesic ball $\mathcal{B}$ to be much smaller than the local curvature scale ($\ell \ll L_{\text{curv}}$), a leading order expansion in their ration can be taken.

There is a unique conformal isometry that preserves the diamond. To see this, let us consider the Minkowski line element in double null coordinates (u, v) defined as $u = t - r$ and $v = t + r$, namely

$$\mathrm{d}s^2 = -\mathrm{d}u\mathrm{d}v + r^2\mathrm{d}\Omega^2 \; . \tag{13.12}$$

With no loss of generality, the spacelike geodesic ball $\mathcal{B}$ can be chosen to lie on a $t = 0$ surface in such a coordinate system and the diamond consists of the intersection of the regions $u > -\ell$

[10] Using a Noetherian approach based on Wald entropy, in 2018 Parikh and Svesko showed that the irreversible contribution is due to the failure of the Killing field to be exact but only approximate at order $\mathcal{O}(x^{-1})$.

[11] For causal diamonds in flat Minkowski spacetime we will have true conformal Killing vector fields preserving the diamond and the associated surface gravity is constant. This is the setting considered in Jacobson's 2016 paper on entanglement equilibrium. Few years later, Visser and Jacobson (2018) extended the analysis to causal diamonds in maximally symmetric spacetimes (MSS) like dS and AdS spacetimes. Causal diamonds in MSS admit approximate conformal Killing vector fields and surface gravity is approximately constant at leading order in a Riemann normal coordinate expansion. For simplicity of exposition and save a bit of time, in this section we will mainly focus on the first case of causal diamonds in flat Minkowski spacetime and only sketch what are the main differences with CDs in MSS.

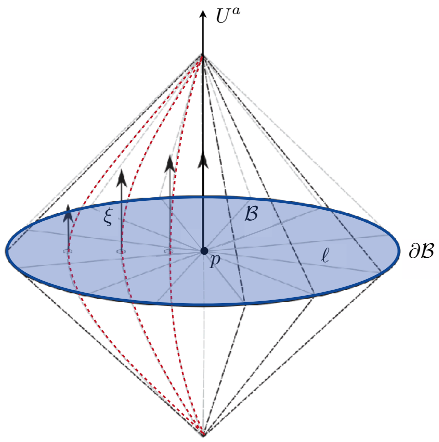


Figure 13.3: A geodesic ball $\mathcal{B}$ of radius ℓ around a point p in a maximally symmetric spacetime. The causal diamond associated with the geodesic ball $\mathcal{B}$ is given by the union of the past and future domain of dependence of $\mathcal{B}$. The red dashed curves represent the flow lines of the conformal Killing vector field ξ, whose flow preserves the diamond and which vanishes at the top and bottom vertices and on $\partial\mathcal{B}$. The latter behaves as a bifurcation surface and the null boundaries of the diamond meeting at $\partial\mathcal{B}$ come to be a conformal Killing horizon for the vector field ξ.

and $v < +\ell$. Any spherically symmetric (in this coordinate system) vector field

$$\xi = A(u)\partial_u + B(v)\partial_v \ , \tag{13.13}$$

is a conformal isometry of the $dudv$ part of the metric, i.e.

$$\mathcal{L}_\xi \mathrm{d}u\mathrm{d}v = \left[A'(u) + B'(v)\right] \mathrm{d}u\mathrm{d}v \ , \tag{13.14}$$

with primes denoting derivatives of the coefficient functions w.r.t. their coordinate arguments. However, the vector field (13.13) is not a conformal isometry of the other piece of the line element as

$$\mathcal{L}_\xi r^2 = \frac{B - A}{r} \ , \tag{13.15}$$

where we used the relation $r = (v-u)/2$. Therefore, in order for the vector field ξ to be a conformal Killing vector field (CKVF) of Minkowski, we need to require that $\mathcal{L}_\xi r^2$ given in (13.15) yields the same conformal factor of the other piece (13.14), thus yielding the following condition for the coefficient functions A, B

$$B(v) - A(u) = \frac{v-u}{2}\left[A'(u) + B'(v)\right] \ . \tag{13.16}$$

Now, at $u = v$ this implies that $B(v) = A(v)$, hence at $v = 0$ the above condition becomes

$$A(u) - A(0) = \frac{u}{2}\left[A'(u) + A'(0)\right] \ , \tag{13.17}$$

whose general solution is given by

$$A(u) = B(u) = a + bu + cu^2 \ . \tag{13.18}$$

The vector field ξ with such A, B generates the conformal isometries of Minkowski spacetime and close a $\mathfrak{sl}(2,\mathbb{R})$ algebra (the spherically symmetric components of Minkowski conformal isometry group which is picked up in the chosen double null coordinate system). But these are not yet the conformal isometries of our causal diamond in Minkowski space. In order to preserve the diamond, the flow of ξ must leave the boundaries $u = -\ell$ and $v = +\ell$ invariant, i.e. $A(\pm\ell) = 0$. The latter provides us with further conditions to determine the unknown constants in (13.18). Using this to express b, c in terms of a, we are left with $A(u) = a(1 - u^2/\ell^2)$ which is unique up to the scaling constant a. We can fix it by normalising the vector field ξ as

$$\xi = \frac{1}{2\ell}\left[\left(\ell^2 - u^2\right)\partial_u + \left(\ell^2 - v^2\right)\partial_v\right] , \tag{13.19}$$

or, in (t, r)-coordinates

$$\xi = \frac{1}{2\ell}\left[\left(\ell^2 - r^2 - t^2\right)\partial_t - 2rt\partial_r\right] , \tag{13.20}$$

which amounts to have unit surface gravity. Indeed, computing the Lie derivative of Minkowski metric along the vector field (13.20), we have

$$\mathcal{L}_\xi \eta_{ab} = -\left(\frac{2t}{\ell}\right)\eta_{ab} . \tag{13.21}$$

The vector field ξ is then not only tangent to the boundary of the diamond as it must be for the flow generated by it to preserve the diamond, but becomes also a true Killing vector on the $t = 0$ surface $\mathcal{B}$. The edge $\partial\mathcal{B}$ of the diamond, where the null generators of the past and future null boundaries of the diamond meet, behaves then as a true bifurcation surface for the conformal Killing vector ξ and the diamond comes then to be a conformal Killing horizon (CKH). Conformal Killing horizons have well-defined surface gravity κ given by $\nabla_a\xi^2 = -2\kappa\xi^a$ and, for ξ normalised as in Eq. (13.20) we have $\kappa = 1$.

13.3.2 The first law of causal diamonds

As it was the case for local horizons in Jacobson's first derivation, before moving to the derivation of the gravitational field equations from thermodynamic equilibrium, we need to identify the relevant geometric quantities for the gravitational system of interest (causal diamonds) and assign to them a thermodynamic interpretation in the context of some nearly equilibrium process involving energy-flux across the surface of the diamond. The relation between geometry and matter energy flux which will allow use to assign the relevant thermodynamic quantities to the system is encodend in the so-called *first law of causal diamonds*.

The starting point is to invoke the covariant phase space on-shell variational identity

$$\delta H_\xi = \int_{\partial\Sigma} \delta Q_\xi - \iota_\xi\theta , \tag{13.22}$$

relating the variation of the Hamiltonian function of the vector field ξ on spacetime to the variation of the Noether charge $(D-2)$-form and the symplectic potential for a generic theory of gravity (and matter) described by a diffeomorphism-covariant Lagrangian D-form. The above identity has been used in lecture 11 to derive the first law of black hole mechanics using the fact that $\delta H_\xi = I_\xi\Omega_\Sigma = 0$ for a symmetry of the metric and matter fields – as it is the case for a true Killing vector of the global bifurcate black hole horizon – and taking Σ to be a hypersurface bounded by the black hole horizon and (spatial) infinity. Now, for the case of causal diamonds considered here, Σ is the ball $\mathcal{B}$, ξ is the conformal Killing vector generating the spherically symmetric conformal isometries of our causal diamond in Minkowski spacetime[12]. Since ξ vanishes at the edge $\partial\mathcal{B}$ of the diamond, the second term on the r.h.s. of Eq. (13.22) identically vanishes and we are left with a single boundary integral

$$\delta H_\xi = \int_{\partial\mathcal{B}} \delta Q_\xi \, . \tag{13.23}$$

The integral on the r.h.s. yields

$$\int_{\partial\mathcal{B}} \delta Q_\xi = -\frac{\kappa}{2\pi}\delta S_{\text{Wald}} = -\frac{\kappa}{8\pi G}\delta\mathcal{A} \, , \tag{13.24}$$

where S_{Wald} is the Wald's entropy, whose explicit form depends on the specific theory of gravity we are dealing with, and $\mathcal{A}$ denotes the area of a spatial section of the causal diamond. The minus sign in the above expression is due to the fact that, according to Stokes' theorem, the positive orientation here is chosen to be outwards, towards larger radius, while the boundary is approached from the interior of the diamond. In the black hole setting considered in lecture 11, instead, the orientation of the Noether charge surface integral on the horizon was chosen to be towards spatial infinity.

Unlike the case of a true Killing vector discussed in the BH case, if ξ is a conformal Killing vector, the Hamiltonian variation δH_ξ on the l.h.s. of (13.23) does not vanish and, in general, it will receive contributions both from matter fields and the gravitational field[13], say

$$\delta H_\xi = \delta H_\xi^{(g)} + \delta H_\xi^{(m)} \, , \tag{13.25}$$

where the superscripts g, m refer to the gravitational and matter contribution, respectively. The gravitational contribution $\delta H_\xi^{(g)}$ is proportional to minus the variation δV of the volume of $\mathcal{B}$, e.g. for the case of general relativity and causal diamonds a background solution given by Minkowski spacetime

$$\delta H_\xi^{(g)} = -\frac{(D-2)\kappa}{8\pi G\ell}\delta V \, , \tag{13.26}$$

where we restored κ that was normalised to unity in the expression of ξ. An explicit proof of the above statement can be found in Appendix D of Jacobson's 2016 paper for the case in which the theory of gravity is GR and the background solution is Minkowski spacetime. The extension to MSS spacetimes is instead discussed in Sec. 3.2 of Jacobson-Visser 2018 paper (see the list of references at the end of this lecture).

Putting everything back together in (13.23), we get

$$-\frac{\kappa}{8\pi G}\left(\delta\mathcal{A} - \frac{(D-2)}{\ell}\delta V\right) = \delta H_\xi^{(m)} \, . \tag{13.27}$$

[12] Actually, causal diamonds in any conformally flat solution to a diffeomorphism-invariant theory of gravity also admit a unique CKVF whose flow preserves the diamond. Moreover, the diamond still have a reflection symmetry around $\mathcal{B}$ so that $\mathcal{L}_\xi g_{ab}\big|_{\mathcal{B}} = 0$.

[13] $\delta H_\xi \neq 0$ also when ξ is a true KVF for the metric but the matter content of the theory does not share the symmetry of the metric. This is the case for instance in presence of a cosmological constant, the latter contributing to the matter stress-energy tensor. For example, when the diamond is the whole static patch of de Sitter space, Eq. (13.23) recovers the first law of dS horizon.

The quantity inside the square bracket on the l.h.s. is nothing but the area variation at fixed volume and, following the thermodynamic notation with subscript quantities kept fixed in the variation, it will be denoted by $\delta\mathcal{A}\big|_V$. The matter contribution on the r.h.s. is given by the conformal boost-energy flux of the matter

$$\delta H_\xi^{(m)} = \int_{\mathcal{B}} \mathrm{d}\Sigma_a T^{ab}\xi_b \,, \tag{13.28}$$

as can be shown from the on-shell relation between the symplectic and Noether current of the matter fields. This will in general receive contributions from all matter fields (also the cosmological constant in the case of dS and AdS spacetimes) through the energy-momentum tensor. This leads us to the first law of causal diamonds

$$\delta H_\xi^{(m)} = \int_{\mathcal{B}} \mathrm{d}\Sigma_a T^{ab}\xi_b = -\frac{\kappa}{8\pi G}\,\delta\mathcal{A}\Big|_V \,, \tag{13.29}$$

according to which, the addition of matter energy to the diamond decreases the area of the boundary at fixed volume. Similarly to the case of local causal horizons discussed in the first part of the lecture, we can now rephrase such a geometric relation in terms of a thermodynamic process for our causal diamond.

13.3.3 Gravitational field equations from causal diamond thermodynamics

Let us consider the past of the causal diamond, i.e. the bottom half below the $t = 0$ codimension-2 spherical surface $\partial\mathcal{B}$ (cfr. set-up of local horizons in Fig. 13.2). We shall now proceed to assign thermodynamic quantities to such a system subject to the process of some matter energy-flux entering the past region of the diamond and compare then the entropy between a time slice at $t = -\epsilon$, with $\epsilon > 0$, and $t = 0$. As discussed in the previous subsections, at the boundary $\xi^2 = 0$ (null) and the boundary of the causal diamond in Minkowki spacetime represents a conformal Killing horizon of constant surface gravity κ[14]. As such, it can be thought of as an isothermal surface with temperature T proportional to κ. Note that:

- Unlike the BH case where the $\kappa\delta\mathcal{A}$ term in the first law of BH mechanics is identified with $T_H\delta S_{BH}$, where $S_{BH} = \frac{\mathcal{A}}{4\hbar G}$ is the Bekenstein-Hawking entropy and $T_H = \frac{\hbar\kappa}{2\pi}$ is the Hawking temperature (cfr. Lectures 11 and 12), the thermodynamic interpretation of the first law (13.29) of causal diamonds calls for a negative temperature $T = -T_H$. Such a negative temperature is not so weird after all. It is already known that the static patch of de Sitter spacetime admits a negative temperature and the first law of dS horizon can be obtained as a limiting case of the first law of causal diamonds in MSS (see Jacobson 2016, Jacobson-Visser 2018). Further details can be found in Sec. 4.1 of Jacobson-Visser paper as as as further arguments and references for assigning a negative temperature to (all) causal diamonds based on the holographic principle and covariant entropy bound. This being said, the first law (13.29) of CD can be written as a standard thermodynamic relation $\delta H_\xi = T\delta S$ between the matter energy variation, temperature, and entropy variation.

[14] In arbitrary spacetimes, there would be curvature corrections so $\xi^2 \sim 0$ only at leading order in a RNC expansion and κ remains approximately constant if the time scale of the process is much smaller than the radius of the geodesic ball, namely $\epsilon \ll \ell$.

- In the above discussion a non-trivial assumption of “local holography” for the gravitational entropy has been made. That is, we associate the entropy to the conformal Killing horizon and assume that this gravitational entropy can also be attributed locally to the spatial sections of the causal diamond whose structure is preserved by the flow of ξ.

Last thing we need to compute is the entropy variation between time $t = 0$ and $t = -\epsilon$, for some $0 < \epsilon \ll \ell$. To this aim, it is convenient to work with a so-called time-like *stretched horizon*, i.e. a codimension-1 time-like surface, rather then directly with our conformal Killing horizon which is a null hypersurface, and then take the limit in which the stretched horizon coincides with the conformal Killing horizon at the the end. We will denote the time-like stretched horizon by Σ. Recalling the expression of the Wald's entropy from lecture 12, the entropy at time t is given by

$$S_{\text{Wald}}(t) = -\frac{1}{4\kappa G}\int_{\partial\mathcal{B}(t)} \mathrm{d}S_{ab}\,\left(P^{abcd}\nabla_c\xi_d - 2\xi_d\nabla_c P^{abcd}\right) \ , \tag{13.30}$$

where $\mathrm{d}S_{ab} = N_{[a}U_{b]}\mathrm{d}\mathcal{A}$ is the infinitesimal binormal element to the codimension-2 surface $\partial\mathcal{B}$, with N_a and U_b respectively the space-like and time-like normals to $\partial\mathcal{B}$, and $P^{abcd} = \frac{\partial L}{\partial R_{abcd}}$ the derivative of the Lagrangian w.r.t. the Riemann tensor and as such it shares the same index symmetries of the latter. Using Stokes' theorem for the antisymmetric (in ab indices) tensor field in the round bracket at the integrand of (13.30), the total change in the Wald entropy in the time interval $t \in [-\epsilon, 0]$ is given by

$$\begin{aligned}\Delta S_{\text{Wald}} &= S_{\text{Wald}}(0) - S_{\text{Wald}}(-\epsilon)\\ &= -\frac{1}{4\kappa G}\int_\Sigma \mathrm{d}\Sigma_a\,\nabla_b\left(P^{abcd}\nabla_c\xi_d - 2\xi_d\nabla_c P^{abcd}\right)\\ &= -\frac{1}{4\kappa G}\int_\Sigma \mathrm{d}\Sigma_a\Big(-\nabla_b\left(P^{abcd} + P^{acbd}\right)\nabla_c\xi_d + P^{abcd}\nabla_b\nabla_c\xi_d - 2\xi_d\nabla_b\nabla_c P^{abcd}\Big) \ .\end{aligned} \tag{13.31}$$

Now, the conformal Killing equation

$$\nabla_a\xi_b + \nabla_b\xi_a = 2\varpi g_{ab} \qquad , \qquad \varpi = \frac{1}{D}\nabla_c\xi^c \tag{13.32}$$

allows us to rewrite the first term in the last line of (13.31) as

$$-\nabla_b\left(P^{abcd} + P^{acbd}\right)\nabla_c\xi_d = \nabla_b P^{adbc}(\nabla_c\xi_d + \nabla_d\xi_c) = 2\varpi g_{cd}\nabla_b P^{adbc} \ . \tag{13.33}$$

The conformal Killing identity

$$\nabla_b\nabla_c\xi_d = R_{ebcd}\xi^e + (\nabla_c\varpi)g_{bd} + (\nabla_b\varpi)g_{cd} - (\nabla_d\varpi)g_{bc} \ , \tag{13.34}$$

on the other hand, allows us to rewrite the second term in the last line of (13.31) as

$$P^{abcd}\nabla_b\nabla_c\xi_d = P^{abcd}R_{ebcd}\xi^e + 2P^{abcd}(\nabla_c\varpi)g_{bd} \ , \tag{13.35}$$

where we used the symmetry properties of the Riemann tensor (hence of P^{abcd}). Inserting the expressions (13.34), (13.35) into the variation of the entropy (13.31), we obtain

$$\Delta S_{\text{Wald}} = -\frac{1}{4\kappa G}\int_\Sigma \mathrm{d}\Sigma_a\left(P^{abcd}R_{ebcd}\xi^e - 2\xi_d\nabla_b\nabla_c P^{abcd} + 2P^{abcd}(\nabla_c\varpi)g_{bd} - 2\varpi g_{cd}\nabla_b P^{adbc}\right) \ , \tag{13.36}$$

and, since the integration is carried over a time-like surface with compact spherical sections, the last two terms integrate to zero at leading order in a RNC expansion for the case of diamonds in a generic MSS spacetime, so that the entropy variation simplifies to

$$\Delta S_{\text{Wald}} \approx -\frac{1}{4\kappa G}\int_\Sigma \mathrm{d}\Sigma_a \left(P^{abcd}R_{ebcd}\xi^e - 2\xi_d\nabla_b\nabla_c P^{abcd}\right) \,. \tag{13.37}$$

The higher-order terms are actually relevant in order to derive the full non-linear gravitational field equations. However, it is possible to show (Parikh and Svesko 2018, Svesko PhD thesis 2020) that, in analogy to the irreversible entropy increase in the free expansion of a gas, the natural increase of our diamond to the past of $t = 0$ can be identified as a process with an associated irreversible entropy increase. The latter turns out to be related to the higher-order contributions coming from the last two terms in Eq. (13.36) and the expression (13.37) can be then identified with the purely reversible contribution ΔS_{rev} to the entropy variation, i.e.

$$\Delta S_{\text{rev}} = -\frac{1}{4\kappa G}\int_\Sigma \mathrm{d}\Sigma_a \left(P^{abcd}R_{ebcd}\,\xi^e - 2\xi_d\nabla_b\nabla_c P^{abcd}\right) \,. \tag{13.38}$$

Finally, by taking Σ to coincide with the conformal Killing horizon of the causal diamond, the interior region becomes causally disconnected from its exterior, thus allowing us to interpret the matter energy-flux (13.28) as heat flow ΔQ which is flowing into macroscopically unobservable degrees of freedom behind a causal barrier. Imposing then Clausius' equality $\Delta Q = T\Delta S_{\text{rev}}$ which relates such a heat-flow with the reversible entropy variation for the present setup, we get the integral equality

$$\int_\Sigma \mathrm{d}\Sigma_a \left(P^{abcd}R_{ebcd}\,\xi^e - 2\xi_d\nabla_b\nabla_c P^{abcd}\right) = 8\pi G\int_\Sigma \mathrm{d}\Sigma_a T^{ab}\xi_b \,. \tag{13.39}$$

As this holds true for any causal diamond around any (non-singular) point in spacetime, the following local relation follows

$$\left(P^{aecd}R_{becd} - 2\nabla_d\nabla_c P^{abcd}\right) N^a\xi^b = 8\pi G\, T^{ab}\, N^a\xi^b \,, \tag{13.40}$$

or equivalently

$$P^{aecd}R_{becd} - 2\nabla_d\nabla_c P^{abcd} + f g_{ab} = 8\pi G\, T^{ab} \,, \tag{13.41}$$

where, similarly to our previous discussion for the case of local Rindler horizons, the term involving the unknown function f arises from the fact that the vector field ξ^b becomes orthogonal to the spacelike unit normal N^a of the codimension-2 boundary surface when the timelike stretched horizon approaches the conformal Killing horizon, that is $g_{ab}N^a\xi^b = 0$. Such a function f can be determined by imposing the covariant conservation of energy-momentum tensor and the contracted Bianchi identity, thus yielding

$$P^{aecd}R_{becd} - 2\nabla_d\nabla_c P^{abcd} - \frac{1}{2}Lg_{ab} + \Lambda g_{ab} = 8\pi G\, T^{ab} \,, \tag{13.42}$$

where $L = L(g^{ab}, R_{abcd})$ and Λ in an integration constant playing the role of a numerically undetermined cosmological constant. The above derivation holds around any non-caustic and non-singular point in spacetime so that the result (13.42) reproduces the field equations for a general diffeomorphism-covariant theory of gravity, which are now obtained as equations of state from (equilibrium) thermodynamics of causal diamonds.

As last comment, let us notice that the whole derivation of the gravitational field equations from spacetime equilibrium thermodynamics discssed in this lecture only involves variation of the geometry due to matter-energy flow, while the quantum state of the matter fields (vacuum) is not changed and, in particular, quantum fluctuations of the energy-momentum tensor T_{ab} are neglected in the thermodynamic limit. In such a limit, the energy-momentum enters the expression of the heat energy-flux through its classical value at p (at least at leading order in RNC expansion) and, in turn, the only contribution to the entropy variation is the one coming from the variation of the geometry with fixed matter field states. This is the so-called UV contribution to the entropy which, in presence of a UV cutoff for our QFT, is proportional to the area. In his second derivation of Einstein's equations in 2016, Jacobson included also the contribution coming from the variation of the state with fixed geometry which yields the so-called IR contribution to the entanglement entropy and, under certain assumptions, he was able to derive the semi-classical Einstein's equation from a generalised equilibrium first law for the total (UV+IR) entanglement entropy. This will be the topic of the next lecture.

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