Laws of BH thermodynamics

Lecture # 12

Last lecture: briefly saw 1^{st} law of BH mechanics

using covariant phase space formalism for classical stationary BH solns

$$\delta E = \frac{\kappa}{2\pi}\delta S + \Omega_H \delta J$$

(mass of BH) asymptotic energy, generating time translations

surface gravity

angular velocity of horizon

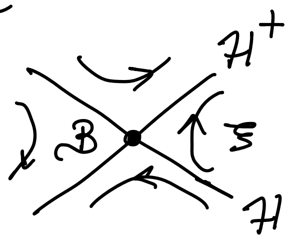
asymptotic angular momentum, generating rotations

Wald entropy

$$S = 2\pi \int_{\mathcal{B}} \tilde{Q}$$

(aptly normalized) charge aspect of KVF $\xi = \partial_t + \Omega_H \partial_\varphi$

Bifurcation surface



For simplicity, here consider Kerr solns in 4d

$\tilde{Q}$: • local geometric qty.

• for a generic diffeo-invariant Lagrangian, can written in terms of

$$\frac{\partial L}{\partial R_{\mu\nu\rho\sigma}}, \quad \nabla_{a_1} \frac{\partial L}{\partial \nabla_{a_1} R_{\mu\nu\rho\sigma}}, \quad \dots, \quad \nabla_{(a_1} \cdots \nabla_{a_m)} \frac{\partial L}{\partial \nabla_{(a_1} \cdots \nabla_{a_m)} R_{\mu\nu\rho\sigma}}$$

But, famous discovery : $S_{BH} \propto$ horizon area

(historically, first not using techniques presented in previous lecture)

For vacuum GR, $L = \sqrt{-g}\, R$, above Wald entropy reduces to

$$S = \frac{A}{4 l_p^2} \qquad \left(l_p^2 = \frac{\hbar G}{c^3} \right)$$

Planck length

$\rightarrow$ Bekenstein-Hawking entropy

Originally proposed by Bekenstein :
(based on Hawking's area law,
and before that, Christodoulou (et. al) 's
irreducible mass results)

$$S = \eta \frac{A}{l_p^2}$$

(universal) real constant

Key insight : S is the physical
entropy of a BH,
encoding our "lack of
knowledge" about the
full details of the
system [thanks to Jaynes]

η : • Bekenstein expected η to be determined
from detailed knowledge of underlying
fundamental QG description

- But was fixed soon after

$$\eta = 1/4$$

Hawking radiation

(return to all this later)

Vast subject, so in this lecture:

- Discuss details that we think were crucial in establishing the foundations
- Do not work in covariant phase space formalism → only 1st law (as mentioned in previous lecture) has been clarified rigorously in this formalism (to the best of our knowledge)

  → here, aim is to give an introduction to some essential aspects of this topic (more or less as done historically), regardless of the choice of the formalism used to set up the full system.

Plan: → Laws of BH mechanics
→ Why and how we can understand these laws as genuine TD laws and BHs as actual TD systems

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Since 1^{st} law was derived in generality in previous lecture, we will not derive it again using "traditional"/historically used methods. Simply state, for now, and return to it later for interpretation:

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J + \Phi_H \delta Q$$

↑ perturbation b/w stationary BHs.

cf. $\delta E = TdS - pdV + \mu dN$

$M \longleftrightarrow E$ (know that really true)

$TdS \longleftrightarrow \frac{\kappa}{8\pi}\delta A$ ... formal analogy (for now)

For 2^{nd} law: need to set up some preliminaries, for properties of Killing horizons

Geodesic Congruences

[Defn.] 1-parameter family of geodesics

[Recall, p49 onward]

$$\gamma : I_1 \times I_2 \longrightarrow M \qquad (I_1, I_2 \overset{\text{open}}{\subset} \mathbb{R})$$
$$s, \lambda \longmapsto \gamma(s, \lambda)$$

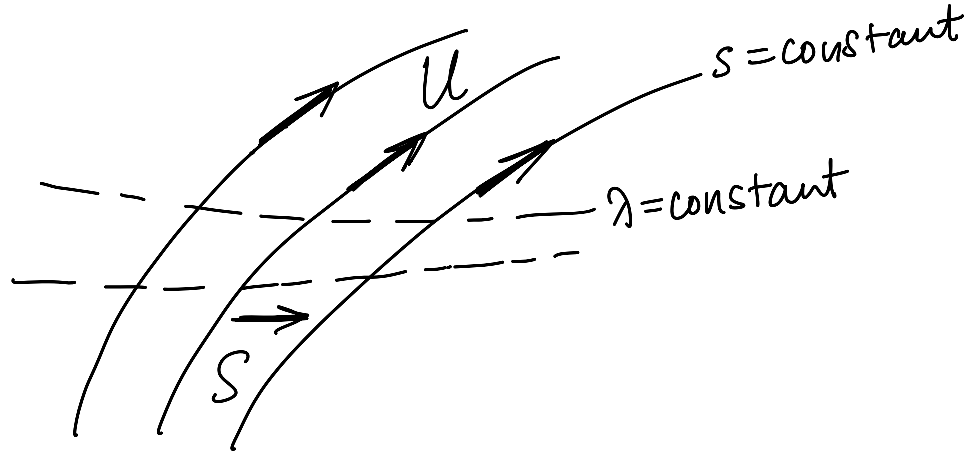
s: labels the geodesic

λ: affine parameter on geodesic γ for fixed s

In some coords.

$S = \partial/\partial s$

$U = \partial/\partial\lambda$



$$[S, U] = 0 \iff \nabla_U S^\mu = \nabla_S U^\mu$$
$$= S^\nu \nabla_\nu U^\mu$$
$$=: B^\mu{}_\nu S^\nu$$

deviation vector (taking b/w nearby geodesics)

$$B^\mu{}_\nu := \nabla_\nu U^\mu$$

tangent vector to the geodesics

quantifies by how much the deviation/displacement S fails to be parallely-transported along γ's

i.e. how much the deviation "evolves" or changes as we move along a geodesic.

$$\nabla_U S = B \cdot S$$

$$B^\mu{}_\nu U^\nu = \nabla_U U^\mu = 0$$

$$B^\mu{}_\nu U_\mu = \tfrac{1}{2} \nabla_\nu (U^2) = 0$$

choice of const. normalisation $U^2 \in \{0, 1, -1\}$

Geodesic Congruence:

family of geodesics s.t. exactly one geodesic passes through each pt.

Here: interested in

Null geodesic Cong.

↳ each geodesic in the congruence is null.

Notice: $\nabla_U (U^\mu S_\mu) = 0$

i.e. $U \cdot S$ is constant along any geodesic in the congruence.

But ambiguity in S:

affine param not unique

$\rightarrow$ can shift it by a constant

$\lambda \mapsto \lambda - a$

In fact, this shift can be different for different geodesics i.e. $a = a(s)$

Thus,

$$S' = S + \tilde{a}(s)\, U \qquad \left(\tilde{a}(s) = \frac{da(s)}{ds} \right)$$

both take us to / displace to (starting from a given geodesic) the <u>same</u> nearby geodesic

Now, notice $U \cdot S' = U \cdot S + \tilde{a}(s)\, U^2$

Spacelike, Timelike: $U^2 \neq 0 \longrightarrow$ ambiguity resolved by choice of $\tilde{a}(s)$

e.g. choose $\tilde{a}$ s.t. $U \cdot S = 0$ @ some pt. on each geodesic (i.e. for every s, choose $\tilde{a}(s)$ s.t. $U \cdot S = 0$ for some λ)

Then $\because$ $U \cdot S = \text{const}$ along λ (for a given s) $\Rightarrow$ $U \cdot S = 0$ everywhere.

While, $U \cdot S'$ remains non-zero.

Null: $U^2 = 0 \longrightarrow$ need a different way to fix ambiguity

$\Rightarrow U \cdot S = U \cdot S'$

Issue: when U is null, the 3d space of vectors orthogonal to U now includes U itself $\Rightarrow$ deviation vectors S orthogonal to U now only specify a 2-parameter family of geodesics $\longleftrightarrow$ missing one dimension

$\rightarrow$ Introduce N^μ (not orthogonal to U) to reach these other null geodesics in the cong.

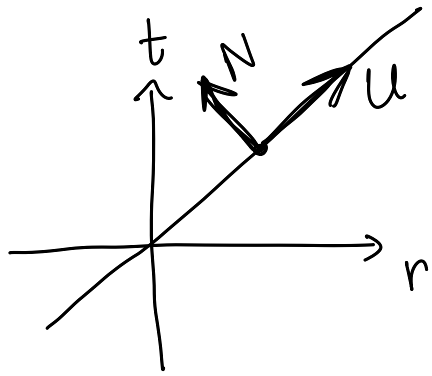
Null geodesic congruences

Construct a vector field N^μ s.t.

$$N^2 = 0 \quad , \quad U \cdot N = -1 \quad , \quad \nabla_U N^\mu = 0$$

i.e. null

if U tangent to outgoing radial null geodesic, then N tangent to an ingoing one



//-transported along the geodesics

Can decompose any S uniquely as:

ambiguity fixed with N

$$S^\mu = a U^\mu + \hat{S}^\mu + b N^\mu$$

where $\hat{S}^\mu$ is s.t. $U \cdot \hat{S} = 0 = N \cdot \hat{S}$

(orthogonal to both)

part orthogonal to U

$\because$ $U^2 = 0$, $U \cdot \hat{S} = 0$

(but: $\nabla_U \hat{S} = \nabla_U S = B \cdot S \neq 0$

not //-transported)

part //-transported along each γ

$\because$ $\nabla_U N^\mu = 0$

(but: $U \cdot N = -1 \neq 0$

not orthogonal to U)

Can write $\hat{S}^\mu = P^\mu_{\ \nu} S^\nu$

$\uparrow$

Projector $P^\mu_{\ \nu} = \delta^\mu_\nu + N^\mu U_\nu + U^\mu N_\nu$

$(P^2 = P)$ ✓

projects onto 2d vector space @ p, $T_p^\perp$ orthogonal to U <u>and</u> N, spanned by $\hat{S}$.

$\nabla_U P^\mu_{\ \nu} = 0$ ∥-transported along U $\because$ both N and U are.

Now: $P\eta = \eta \implies \nabla_U \eta^\mu = \hat{B}^\mu_{\ \nu} \eta^\nu$

(i.e. any $\eta \in T^\perp$)

where $\hat{B}^\mu_{\ \nu} = P^\mu_{\ \rho} B^\rho_{\ \sigma} P^\sigma_{\ \nu}$

i.e. any $\eta \in T^\perp$ remains in this subspace under evolution along the geodesics.

Interested in BH horizons : null hypersurfaces $\mathcal{N}$

$\rightarrow$ to understand behaviour of $\mathcal{N}$, must understand behav. of (null) congruence of <u>generators</u> of $\mathcal{N}$

i.e. consider null congruences
on a null hyp. $\mathcal{N}$

$\rightarrow$ S tangent to $\mathcal{N}$

i.e. $U \cdot S = 0 \iff b = 0$

($\because$ U null and normal to $\mathcal{N}$)

$$\Rightarrow S^\mu = a U^\mu + \hat{S}^\mu$$

Also, $U \cdot S = 0 \Rightarrow \nabla_U \hat{S}^\mu = \hat{B}^\mu{}_\nu \hat{S}^\nu$

$\hat{B}^\mu{}_\nu$ tensor on $T_p^\perp$ (2d)

$\rightarrow$ 2x2 matrix : decompose

trace	traceless symm.	traceless antisymm.
θ	$\hat{\sigma}_{\mu\nu}$	$\hat{\omega}_{\mu\nu}$
expansion	shear	rotation

$\theta = \hat{B}^{\mu}{}_{\mu}$, $\hat{\sigma}_{\mu\nu} = \hat{B}_{(\mu\nu)} - \frac{1}{2} P_{\mu\nu} \theta$

$\uparrow$ $\hat{\omega}_{\mu\nu} = \hat{B}_{[\mu\nu]}$

interested in this

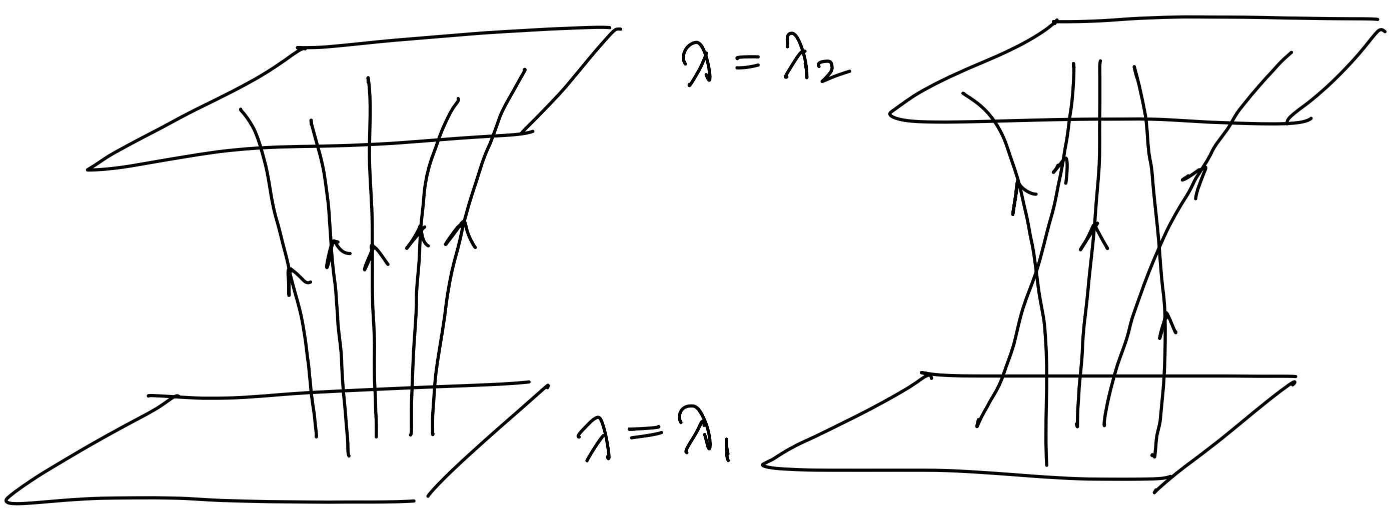
i.e. $\hat{B}^{\mu}{}_{\nu} = \frac{1}{2} \theta P^{\mu}{}_{\nu} + \hat{\sigma}^{\mu}{}_{\nu} + \hat{\omega}^{\mu}{}_{\nu}$

Back to

For null congruences on $\mathcal{N}$: $\hat{\omega}_{\mu\nu} = 0$ on $\mathcal{N}$

In fact, U normal to a family of null hypersurfaces (i.e. λ = const.) $\Longleftrightarrow$ $\hat{\omega} = 0$.

[Townsend, p. 105]



expansion shear

expansion : neighbouring geodesics move apart ($\theta > 0$)
or closer together ($\theta < 0$) as "evolve in λ"

shear : neighbouring geodesics move apart in one direction, and closer in orthogonal direction, keeping constant cross-section area

∵ traceless

Raychaudhuri's eqn (for null congruences)

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}^2 + \hat{\omega}^2 - R_{\mu\nu} U^\mu U^\nu$$

For null hypersurfaces $\mathcal{N}$:

For $\mathcal{N}$ Killing horizon of KVF ξ $\qquad \hat{B}_{\mu\nu} = 0$ and $\dfrac{d\theta}{d\lambda} = 0$

$$\Rightarrow R_{\mu\nu}\xi^\mu \xi^\nu \Big|_{\mathcal{N}} = 0$$

Also: If $\theta = \theta_0 < 0$ @ some pt. $p \in \gamma$ on a null generator of $\mathcal{N}$, then $\theta \to -\infty$ in finite affine length.

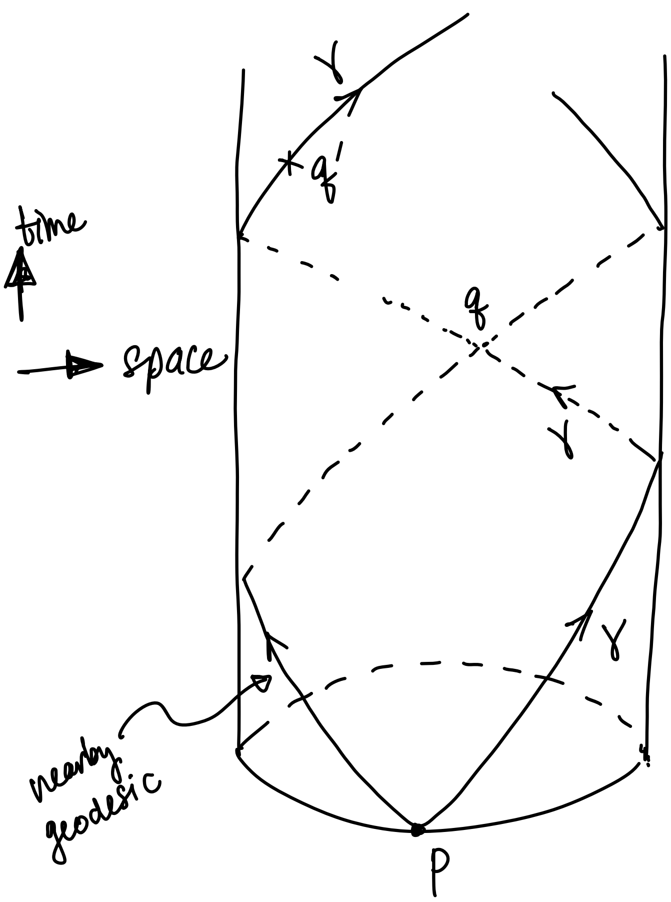
(assuming Null Energy Condition and Einstein's eqn)

In other words, if $\theta < 0$ @ some pt. p on γ (on $\mathcal{N}$), then the geodesics must converge to a "focus" or "caustic" @ q $\longrightarrow$ Conjugate pts. p and q

E.g. Light rays in flat 2d cyclindrical spacetime $\mathbb{R} \times S^2$

points p and q conjugate

$\rightarrow$ any pt q' on γ, in future of q, is timelike separated from p



(Defn) Points p and q on a geodesic γ are <u>Conjugate</u> if $\exists$ a Jacobi field (i.e. a soln. of the geodesic deviation eqn.) along γ that vanishes @ p and q but is not identically zero.

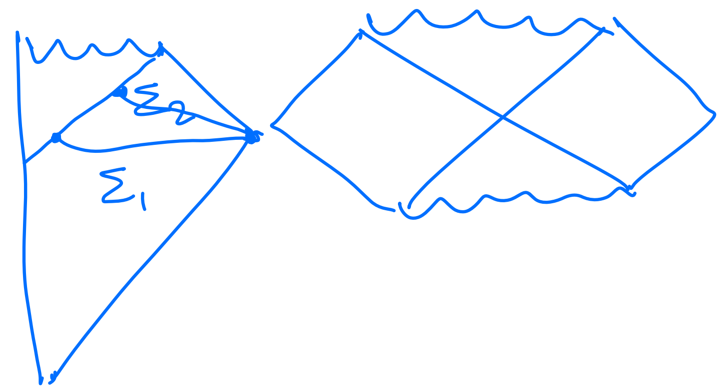
If p and q are conjugate, then $\exists$ multiple infinitesimally nearby geodesics that pass through p <u>and</u> q.

Area law

Consider a strongly asymptotically predictable spacetime satisfying EEqns. with null energy condition.

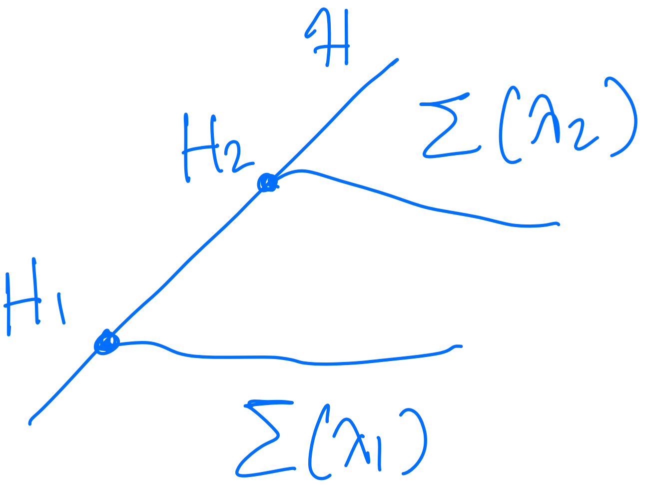
basically means that $\exists$ a globally hyperbolic submfld. $\mathcal{U} \subset M$ containing both the exterior spacetime and horizon e.g.

(i.e. singularity not naked and "predictable" spacetime accessible)



$\Longrightarrow$ $\exists$ a family of Cauchy hypersurfaces $\Sigma(\lambda)$ for $\mathcal{U}$ s.t. $\Sigma(\lambda_2) \subset D^+(\Sigma(\lambda_1))$

if $\lambda_2 > \lambda_1$.



"Areas of horizons"

Area(H_1), Area(H_2)

$$H_i = \mathcal{H} \cap \Sigma_i$$

NEC: $T_{\mu\nu} \ell^\mu \ell^\nu \geq 0 \quad \forall$ null ℓ^μ.

Then: $\text{Area}(H_2) \geq \text{Area}(H_1)$

2^{nd} law of BH mech. [Hawking's area theorem]

Sketch of proof

By contradiction, to show that $\theta \geq 0$ on $\mathcal{H}$.

Let $\theta < 0$ on $\mathcal{H} \Rightarrow$ rays must focus

$\Rightarrow \exists$ conjugate pts. on rays

The existence of a conjugate pt. to the future of a null generator of $\mathcal{H}$ means that this generator leaves $\mathcal{H}$.

But, it cannot [thm. by Penrose] (no future endpoints on generators of $\mathcal{H}$)

$\Rightarrow \theta \geq 0$ everywhere on $\mathcal{H}$

$\rightarrow \quad \delta A \geq 0$.

Remark: $\theta = 0$ for stationary BH

Zeroth law

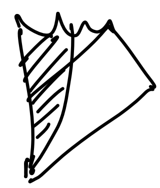
Couple of lectures back:

$$\nabla_\xi \kappa^2 = 0$$ for κ surface gravity of KH $\mathcal{N}$.

i.e. κ constant on each orbit of ξ

$\rightarrow$ but can it change value b/w diff. orbits?

Yes, for bifurcate KHs.

But what about KHs in general eg. 

for BH as a result of collapse?

Zeroth law: In a stationary BH spacetime satisfying the Dominant Energy Condition, κ is constant on the event horizon.

DEC: $T_{\mu\nu} V^\mu V^\nu \geqslant 0$ and $T^\mu{}_\nu V^\nu$ causal

for all timelike V^μ.

<u>Proof</u>: Stationary BH
Event horizon $\mathcal{H}$: Killing horizon of some KVF ξ

Then, on $\mathcal{H}$:

$$0 \underset{\text{Raycha}}{=} R_{\mu\nu} \xi^\mu \xi^\nu \Big|_{\mathcal{H}} \overset{\text{EEqn. and } \xi^2|_{\mathcal{H}}=0}{=} T_{\mu\nu} \xi^\mu \xi^\nu \Big|_{\mathcal{H}}$$

$$=: C_\mu \xi^\mu \Big|_{\mathcal{H}}$$

$\Rightarrow C^\mu = T^\mu{}_\nu \xi^\nu$ tangent to $\mathcal{H}$

$\Rightarrow$ expand: $C^\mu = a\xi + b_1 \eta_1 + b_2 \eta_2$

span $T^\perp$

$C^2 \geqslant 0$ on $\mathcal{H}$ $\quad$ $\because$ $\xi \cdot \eta \big|_{\mathcal{H}} = 0$, $\xi^2 \big|_{\mathcal{H}} = 0$

But DEC $\Rightarrow$ $C^2 \leq 0$

$\Rightarrow C^2 = 0$ null

$\Rightarrow C \propto \xi$ on $\mathcal{H}$

i.e. $\xi_{[\mu} T_{\nu]\rho} \xi^\rho \big|_{\mathcal{H}} = 0$

$\Rightarrow$ $\xi_{[\mu} R_{\nu]\rho} \xi^\rho = 0$

EEqn.

$\Rightarrow \xi_{[\mu} \partial_{\nu]} K = 0$

$\Rightarrow \partial_\mu K \propto \xi_\mu$

Thus: $\nabla_U K \big|_{\mathcal{H}} = 0$ for any tangent U on $\mathcal{H}$. $\square$

Stationary BHs at "thermal eqm." at constant K.

Third law ~ K cannot be reduced to zero by any physical process (in finite # of steps).

Resemblence of these laws of BH mechanics, to TD laws of material systems, is quite strong, but: (from what we've seen)

- $S \neq A$ on dimensional grounds
  $\uparrow$ dimensionless, $\uparrow$ $[\ell^2]$

- $\delta A \geq 0$ for each BH (separately), but in TD $\delta S_{\underline{\underline{total}}} \geq 0$

Bekenstein's proposal

- BH is "black": perfect absorber $T_{BH} = 0$

Hawking radiation

Bekenstein's proposal: (saw briefly @ the start)

$$S_{BH} = \frac{\eta A}{\hbar G}$$ <u>physical</u> entropy

$(c=1)$ appearance of "QM"

$\longrightarrow$ Generalised 2^{nd} law

$\because$ S_{BH} is a physical entropy and 2^{nd} law of TD must hold, then total entropy must be $S_{tot} = S_{BH} + S_{ext}$

ordinary entropy of outside matter

$$\delta\left(S_{BH} + S_{ext}\right) \geqslant 0 .$$

$\hbar \neq 0$ in S_{BH} : must include quantum theory !

Indeed: Classically $T_{BH} = 0$, $S_{BH} = $ "∞"

QM $T_{BH} \neq 0$ (but tiny), $S_{BH} < \infty$ (but large)

Confirmed by Hawking (radiation)

Interesting to note how QM came to the rescue even earlier for solving an issue in SM / TD: UV catastrophe
$\longrightarrow$ birth of QM (by Planck, etc.)

Only after "including QM", do we get a physical interpretation of above laws as genuine laws of TD, applied to BHs.

BH temperature (very quick)

— detailed (original) treatment needs QFT on curved spacetimes
— more modern: algebraic treatments

Here: skip details, only give an idea.

A defining property of thermal eqm. states : KMS condition

—— periodicity in 2-pt. correlators

Consider

$$\rho_\beta = e^{-\beta H} \qquad (\text{eqm. @ temp } \beta^{-1})$$

Then : for any observables $A, B \in \mathcal{B}(\mathcal{H})$

$$\langle A\, e^{-iH(t+i\beta)} B\, e^{iH(t+i\beta)} \rangle_{\rho_\beta} = \langle e^{-iHt} B\, e^{iHt} A \rangle_{\rho_\beta}$$

Periodicity : $\hbar\beta$ in "imaginary time"

KMS condition $\Longleftrightarrow$ State is @ thermal eqm.

Consider , Schwarzschild :

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\Omega^2$$

Wick rotate $t \mapsto i\tau$ to imaginary time

Euclidean Schw:

$$ds_E^2 = \left(1-\frac{2M}{r}\right)d\tau^2 + \frac{dr^2}{\left(1-\frac{2M}{r}\right)} + r^2 d\Omega^2$$

Coord. singularity @ $r=2M$

Near $r=2M$: $r-2M =: \frac{x^2}{8M}$

$$ds_E^2 \approx \underbrace{(Kx)^2 d\tau^2 + dx^2}_{ds_R^2} + \frac{1}{4K^2} d\Omega^2$$

2-dim Rindler!

i.e. Schw. geometry near event horizon looks like Rindler, with acc. $K = 1/4M$

($r > 2M$, $x > 0$: exterior)

$$ds_R^2 = dx^2 + x^2 d(\kappa\tau)^2$$

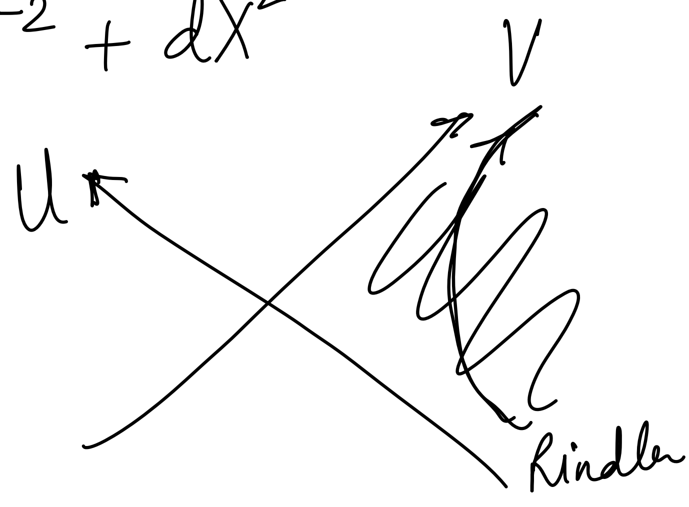
$\rightarrow$ flat 2d Euclidean, polar coords. x, τ

with periodicity $\tau \sim \tau + \dfrac{2\pi}{\kappa}$

If wanted, as before, can introduce null coords etc: $U = -x e^{-\kappa\tau}$, $V = x e^{\kappa\tau}$

$$U = T - X \quad , \quad V = T + X$$

$$ds_R^2 = -dU\,dV = -dT^2 + dX^2$$



Now, let's have a quantum field on Schw

$\Phi(\underline{x}, \tau)$ (Wick rotated)

Dynamics $Z = \int [D\Phi]\, e^{-S_E(\Phi)}$

Euclidean action for Φ

Periodicity in τ from Eucl. Schw. $\Longrightarrow$ Periodicity in τ in $\Phi(\underline{x}, \tau)$ with period $\hbar\beta$

$\longrightarrow$ can write

$$Z = \mathrm{Tr}\left(e^{-\beta H}\right)$$

Hamiltonian for Φ from S_E

β: physical temp.

But know: $\hbar\beta = \dfrac{2\pi}{\kappa}$

$\Rightarrow$ QFT can be in eqm. with a BH only @ Hawking temp. $\quad T_H = \dfrac{\kappa}{2\pi}\dfrac{\hbar}{k_B c}$

Remarkable formula!

- Usually, way to arrive @ this is semi-classically : QM matter
  Classical spacetime

But result evidently independent of matter details

$\longrightarrow$ thermal nature of spacetime itself? [Bekenstein, Padma ...]

Bekenstein : S_{BH} really counts QG microstates of BH (i.e. spacetime)

→ Hints @ underlying quantum, discrete structure of spacetime

- Physical entropy of BH
  — what does it really mean?
  → thermal entropy?
  → entanglement entropy — Quantum info. and Gravity

Led to the idea of

- Holography : ['tHooft, Susskind]

  $S_{BH} \propto A$, not volume

  Hypothesis : info. in a finite region must be completely encoded in its 2d boundary

  ... AdS/CFT, edge modes etc.

- BH evaporation : information loss?

Further Reading

- Bekenstein 1972, "Black holes and the second law"
- Bekenstein 1973, "Black holes and entropy"
- Bardeen, Carter, Hawking 1973, "The four laws of black hole mechanics"
- Townsend, Black holes Lecture notes [gr-qc/9707012]
- Reall, Part 3 Black hole notes
- Jacobson 1996, "Introductory Lectures on Black Hole Thermodynamics"
- Jacobson 2018, "Entropy from Carnot to Bekenstein" [arXiv: 1810.07839]
- Compere, 2006, "An introduction to the mechanics of black holes" [gr-qc/0611129]