10 Energy, angular momentum and electric charge

Last time, we looked at the properties of stationary asymptotically flat four-dimensional spacetimes containing black holes, and saw that they are classified by their symmetries. The Schwarzschild spacetime is spherically symmetric, and is a special case of the Kerr spacetime, which is axisymmetric. If we turn on an electric field, the most general case is given by the Kerr-Newman spacetime, which in the spherically symmetric case is called the Reissner-Nordström spacetime.

Of course, these metrics are more than just solutions to the Einstein or Einstein-Maxwell equations. They describe objects with physical properties like energy, angular momentum and electric charge. For example, the energy of a Schwarzschild black hole is equal to its mass, while it has zero angular momentum and electric charge. But what does it actually mean to say this? It is a little difficult to imagine trying to feel a black hole's angular momentum in the same way you might feel the angular momentum of a spinning bike wheel.

To make progress, we can use a very precise definition of these physical quantities – they are the Hamiltonian generators of certain symmetries. The energy generates time evolution, the angular momentum generates rotation around an axis, and the electric charge generates a global Maxwell gauge transformation. We can use these definitions to obtain precise formulae for these quantities for the various black holes. Our formulae will also apply beyond the exact Kerr-Newman family. Indeed, astrophysical black holes are not in the Kerr-Newman family, since they are only approximately stationary. The metric of such a black hole is instead generically some perturbation of the Kerr-Newman solution, and our formulae will allow us to find the contributions of this perturbation to the black hole's physical properties.

We first need to set up a phase space formalism for treating asymptotically flat black hole spacetimes. We will use the covariant phase space methods that we described in the first part of the course to do this.

10.1 Asymptotically flat spacetime

We will need a slightly more precise definition of asymptotic flatness. There are different possible choices we can make here – which one is most appropriate is still an area of active research. First we'll use a coordinate-free definition, and then we will set up some coordinates that will allow us to proceed with an analysis of the symmetries. Note that this is *very* far from the whole story. We will just discuss the bare essentials.

At first we will ignore the electromagnetic field. Afterwards we will turn it back on and see what changes.

Definition. A spacetime is *asymptotically simple* if it can be conformally compactified, and every null geodesic has 2 endpoints at infinity, i.e. in the boundary of the conformal compactification.

For example, Minkowsi spacetime is asymptotically simple. However, black hole spacetimes are not asymptotically simple, because null geodesics can enter the black hole and never escape. To account for this, we define:

Definition. A *weakly asymptotically simple* spacetime $\mathcal{M}$ is one for which there exists an open set $U \subset \mathcal{M}$ that is conformally isometric to an open neighbourhood of the boundary of $\widetilde{\mathcal{M}}$, where $\widetilde{\mathcal{M}}$ is the conformal compactification of some asymptotically simple manifold.

Definition. An *asymptotically flat* spacetime is one which is weakly asymptotically simple, and which obeys the vacuum Einstein equations $R_{\mu\nu} = 0$ in an open neighbourhood of the boundary, called the *asymptotic region*.

In other words, an asymptotically flat spacetime has no energy-momentum near its conformal boundary.

By this definition, an asymptotically flat spacetime is in fact *exactly* Ricci-flat near infinity. But the curvature of the gravitational field is described by more than just the Ricci curvature – indeed it is described by the full Riemann curvature tensor $R^{\mu}{}_{\nu\rho\sigma}$. It can be shown however that, as one ventures deeper and deeper into the asymptotic region, the full Riemann curvature becomes smaller and smaller. So indeed, an asymptotically flat spacetime is approximately Riemann-flat in the asymptotic region.

A series of very important results due to Penrose, Geroch, Bondi, Sachs, and many others, explained more precisely how $R^{\mu}{}_{\nu\rho\sigma}$ behaves in the asymptotic region. We won't give all the details of these results, but just give a very broad outline.

The conformal boundary of an asymptotically flat spacetime has different components, similar to the exact case. There are future and past null infinity $\mathscr{I}^{\pm}$, which are defined as the subsets of the boundary consisting of all future and past endpoints respectively of null geodesics. And there is spacelike infinity i^0, which is the part of the boundary which is spacelike separated from all points in spacetime. It will be important for us to understand the behaviour of the metric near both $\mathscr{I}^{\pm}$ and i^0.

It actually turns out the analysis is simplest near null infinity. The reason for this boils down to what is known as the 'peeling property' of the Weyl tensor, which is the 'traceless' part of the Riemann tensor, defined as

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - g_{\mu[\rho}R_{\sigma]\nu} + g_{\nu[\rho}R_{\sigma]\mu} + \frac{1}{3}Rg_{\mu[\rho}g_{\sigma]\nu}. \tag{10.1}$$

We will focus on future null infinity, but similar results apply to past null infinity by time reversal. Suppose we have a null geodesic with its endpoint at future null infinity, and let r be an affine parameter along this geodesic. We can evaluate the components of the Weyl tensor along this geodesic. The peeling property schematically then takes the form

$$C = \frac{C_4}{r} + \frac{C_3}{r^2} + \frac{C_2}{r^3} + \frac{C_1}{r^4} + \mathcal{O}\left(\frac{1}{r^5}\right). \tag{10.2}$$

Here C_1, C_2, C_3, C_4 are rank 4 tensors which are independent of r. The number in the subscript indicates some behaviour of the tensor (if you are interested, it gives the multiplicity of the tangent vector to the geodesic as a principal null direction of the tensor). Assuming the metric obeys some analyticity properties, one can show that the peeling property holds in the asymptotic region of an asymptotically flat spacetime.

One can then translate the peeling property and Einstein equations into conditions on the metric. To see what happens, let us first pick some coordinates near null infinity. A particularly useful choice of coordinates is due to Bondi, Van der Burg, Metzner and Sachs (BMS).

Consider exactly flat 4d spacetime, i.e. the Minkowski metric

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu} = -\,\mathrm{d}t^2 + \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x}\,. \tag{10.3}$$

It is useful to change to spatial spherical coordinates r, θ^A and retarded time $u = t - r$. Here, θ^A are some coordinates on the 2-sphere indexed by capital letters $A, B, \ldots$. In these coordinates, let γ_{AB} be the round metric on the unit 2-sphere. The Minkowski metric then takes the form

$$\mathrm{d}s^2 = -\,\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}r + r^2\gamma_{AB}\,\mathrm{d}\theta^A\,\mathrm{d}\theta^B\,. \tag{10.4}$$

There are two things we can note about these coordinates.

- The induced metric on a constant u surface is $r^2\gamma_{AB}\,\mathrm{d}\theta^A\,\mathrm{d}\theta^B$, which is degenerate, so constant u surfaces are null.
- The induced area form on constant u, r surfaces is proportional to r^2. (We sometimes say that r is a 'luminosity distance', because it is compatible with an inverse square law for the intensity of a light source placed at the origin.)

One of the insights of BMS was that, for a general metric $\mathrm{d}s^2$, we can pick coordinates u, r, θ^A such that these conditions continue to hold. In this case θ^A are a set of coordinates transverse to u, r. Since this is a set of coordinates, it is a gravitational gauge choice and is known as Bondi gauge.

One may show that in Bondi gauge a general metric takes the form

$$\mathrm{d}s^2 = -U\,\mathrm{d}u^2 - 2e^{2\beta}\,\mathrm{d}u\,\mathrm{d}r + g_{AB}\Big(\mathrm{d}\theta^A + \frac{1}{2}U^A\,\mathrm{d}u\Big)\Big(\mathrm{d}\theta^B + \frac{1}{2}U^B\,\mathrm{d}u\Big), \tag{10.5}$$

where

$$\partial_r \det\!\Big(g_{AB}/r^2\Big) = 0. \tag{10.6}$$

Here g_{AB}, U^A are a metric and vector field respectively on the transverse spaces of constant u, r, and U, β, g_{AB}, U^A all depend on u, r, θ^A. In the exactly flat case, we had

$$U = 1, \quad \beta = U^A = 0, \quad g_{AB} = r^2\gamma_{AB}. \tag{10.7}$$

The asymptotic region is at large r (far from any matter near the origin). It seems clear that for a metric to be asymptotically flat, the values taken by U, β, U^A, g_{AB} should, in the asymptotic region, be very close to those for the flat metric. Thus we need to impose falloff conditions on U, β, U^A, g_{AB}. The peeling property tells us what these conditions should be.

For simplicity, let us assume that the surfaces of constant u, r are topologically 2-spheres, so that the topology of the asymptotic region is the same as that of Minkowski space. Then it is convenient to use complex coordinates $\theta^A = z, \bar{z}$ on the 2-sphere, which are related to the standard coordinates via

$$z = e^{i\phi}\tan(\theta/2). \tag{10.8}$$

The unit metric on the 2-sphere is then given by $2\gamma_{z\bar{z}}\,\mathrm{d}z\,\mathrm{d}\bar{z}$, where

$$\gamma_{z\bar{z}} = \frac{2}{(1 + z\bar{z})^2}. \tag{10.9}$$

One can compute the Weyl curvature in these coordinates, and impose the peeling property. The result is the set of falloff conditions

$$U = 1 - \mathcal{O}\left(\frac{1}{r}\right), \tag{10.10}$$

$$\beta = \mathcal{O}\left(\frac{1}{r^2}\right), \tag{10.11}$$

$$U^z, U^{\bar{z}} = \mathcal{O}\left(\frac{1}{r^2}\right), \tag{10.12}$$

$$g_{zz}, g_{\bar{z}\bar{z}} = \mathcal{O}(r), \tag{10.13}$$

$$g_{z\bar{z}} = r^2\gamma_{z\bar{z}} + \mathcal{O}(1). \tag{10.14}$$

and at large r, one finds that the metric may be written in the form

$$\begin{aligned} \mathrm{d}s^2 = &- \mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}r + 2r^2\gamma_{z\bar{z}}\,\mathrm{d}z\,\mathrm{d}\bar{z} \\ &+ \frac{2m_B}{r}\,\mathrm{d}u^2 + rC_{zz}\,\mathrm{d}z^2 + rC_{\bar{z}\bar{z}} + \mathrm{D}^z C_{zz}\,\mathrm{d}u\,\mathrm{d}z + \mathrm{D}^{\bar{z}} C_{\bar{z}\bar{z}}\,\mathrm{d}u\,\mathrm{d}\bar{z} \\ &+ \frac{1}{r}\left(\frac{4}{3}(N_z + u\partial_z mB) - \frac{1}{4}\partial_z(C_{zz}C^{zz})\right)\mathrm{d}u\,\mathrm{d}z + \text{complex conjugate} + \dots . \end{aligned} \tag{10.15}$$

The first line is just the Minkowski metric, while the later lines represent corrections to that metric. In these corrections, $m_B, C_{zz}, C_{\bar{z}\bar{z}}, N_z, N_{\bar{z}}$ are functions of $u, z, \bar{z}$, and D is the covariant derivative associated to the round metric on the unit 2-sphere, and we also use that metric to raise and lower $z, \bar{z}$ indices. m_B is known as the Bondi mass aspect, $N_z, N_{\bar{z}}$ as the angular momentum aspects, and $\partial_u C_{zz}, \partial_u C_{\bar{z}\bar{z}}$ as the Bondi news. For the metric to be real, the functions should obey $C_{zz} = (C_{\bar{z}\bar{z}})^*$ and $N_z = (N_{\bar{z}})^*$.

We can also do an expansion of the metric at past null infinity. We won't go into the details here, but suffice it to say that very similar things happen because of the time reversal symmetry.

It will also be important to know how the metric behaves at spatial infinity. On a Penrose diagram, spatial infinity appears to be located at the past of future null infinity, so we might hope to be able to get the behaviour of the metric at spatial infinity by taking $u \to -\infty$. However, the Penrose diagram is a bit misleading in this respect. On it, spatial infinity appears like a single point, but this is only an artifact of the conformal compactification procedure. In the physical uncompactified spacetime, spatial infinity is really a timelike surface of topology $S^2 \times \mathbb{R}$. By going to the past of future null infinity, we only reach the future of spatial infinity. To proceed from there to the entirety of spatial infinity, it is natural to assume that the radial falloff conditions are time-independent.

We'll skip the details, but the result of such an analysis is the following: in a set of 'Cartesian-like' coordinates t, x, y, z near spatial infinity, the metric takes the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ for an $h_{\mu\nu}$ obeying

$$h_{\mu\nu} = \mathcal{O}\left(\frac{1}{r}\right), \quad h_{\mu\nu,\rho} = \mathcal{O}\left(\frac{1}{r^2}\right), \quad h_{\mu\nu,\rho\sigma} = \mathcal{O}\left(\frac{1}{r^3}\right), \tag{10.16}$$

where $r^2 = x^2 + y^2 + z^2$.

The asymptotic falloffs we've described are consistent with all the kinds of physical spacetimes one usually considers, containing black holes, gravitational waves, and so on.

Actually, we're sweeping a lot under the rug here. Deducing the exact relationship between the falloff conditions on null infinity and those on spatial infinity is a *very* difficult mathematical problem. The proofs of properties of this relationship tend to be hundreds of pages long – see for example the famous monograph by Christodoulou and Klainerman. Obviously it's not worth our time at the moment to go into all those details. So it would be more precise to call the behaviour of the metric at spatial infinity given above an *assumption*. Moreover, we *assume* that it is consistent with the more rigorously understood behaviour at null infinity.

10.2 Including electromagnetism

So far what we have said appleis to the case when there is no electromagnetic field, but turning it on leads to very similar things happening. In order to allow for long range electromagnetic effects, we have to modify the definition of asymptotically flat to:

Definition. An *asymptotically flat* spacetime is one which is weakly asymptotically simple, and which obeys the Einstein-Maxwell equations in an open neighbourhood of the boundary.

In this case the peeling property continues to apply for the Weyl tensor. Additionally there is a peeling property that can be shown to hold for the electromagnetic field strength $F_{\mu\nu}$. Using these peeling properties, we can find falloff conditions on the metric and electromagnetic fields. It then turns out that the metric has the same falloff conditions.

For the Maxwell field, one can pick a gauge in which at future null infinity the in Bondi coordinates components of the gauge potential have the falloffs

$$A_u = \mathcal{O}\left(\frac{1}{r}\right), \quad A_r = 0, \quad A_z = A_{\bar{z}} = \mathcal{O}(1). \tag{10.17}$$

Similarly, at spatial infinity one can pick a gauge such that in the Cartesian-like coordinates the components of the gauge potential behave like

$$A_\mu = \mathcal{O}\left(\frac{1}{r}\right), \quad A_{\mu,\nu} = \mathcal{O}\left(\frac{1}{r^2}\right). \tag{10.18}$$

The relationship between these two sets of falloff conditions is at the same level of rigour as for the metric.

10.3 Phase space of asymptotically flat spacetimes

Now that we understand a bit better what an asymptotically flat spacetime is, we are in a better position to construct the phase space of asymptotically flat spacetimes. We will do so using the covariant phase space methods developed earlier in the course.

The phase space consists of all asymptotically flat spacetimes with on-shell field configurations, so the only thing we need to do is find the unambiguous version of the presymplectic form. The Einstein-Maxwell action is $S = \int_{\mathcal{M}} L + \int_{\partial\mathcal{M}} l$, where

$$L = \frac{1}{16\pi G} R\varepsilon - \frac{1}{2e^2} F \wedge *F, \qquad l = \frac{1}{8\pi G} K\varepsilon_\partial. \tag{10.19}$$

We have seen previously what presymplectic potential density comes from the pure Einstein action, and we have seen what it is for the Maxwell action. In the Einstein-Maxwell case, we basically just have to add these two pieces together, and we get

$$\theta = \frac{1}{e^2}\delta A \wedge *F + \iota_V \varepsilon, \quad \text{where } V^\mu = \frac{1}{16\pi G}(g^{\mu\rho}\nabla^\sigma \delta g_{\rho\sigma} - g^{\rho\sigma}\nabla^\mu \delta g_{\rho\sigma}). \tag{10.20}$$

Using this, we can get the presymplectic form. But first, we need to check that it is compatible with our boundary conditions / falloff conditions. As we discussed in detail previously, in the case of finite boundaries to find compatibility it is enough to show that

$$\Big(\theta + \delta l - \mathrm{d}C\Big)\Big|_{\mathcal{I}} = 0, \tag{10.21}$$

for some C.

But asymptotically flat spacetimes are non-compact, and so the boundary is not finite. We need to regularise the construction – recall that this involves restricting spacetime to some compact region, and then taking the limit as that region grows to encompass the whole spacetime. When we were analysing fields in exactly flat Minkowski space, the compact region we considered was obtained by restricting to a finite radius $r < R$. Then we took the limit $r \to \infty$.

In the asymptotically flat case we can do basically the same thing. We restrict to finite $r < R$, where now r is the coordinate near spatial infinity constructed above. Then at the end we take $r \to \infty$.

Consider then the evolution between two Cauchy surfaces Σ_-, Σ_+ in asymptotically flat spacetime. Before taking $r \to \infty$, the boundaries of these Cauchy surfaces $\partial\Sigma_-, \partial\Sigma_+$ lie at $r = R$, and are connected by a timelike surface $\mathcal{I}$ at $r = R$. In this case, the finite boundary compatibility condition (10.21) is replaced by

$$\lim_{r\to 0} \int_{\mathcal{I}} (\theta + \delta l - \mathrm{d}C) = 0 \tag{10.22}$$

for some C. We won't show it explicitly, but if we pick

$$C = \iota_W \varepsilon_\partial, \qquad W^\mu = -\frac{1}{16\pi G}(g^{\mu\nu} - n^\mu n^\nu) n^\rho \delta g_{\nu\rho}, \tag{10.23}$$

where n is the unit normal to $\mathcal{I}$, then, as a direct consequence of the falloff conditions described above, this condition is satisfied.

Thus, the asymptotically flat falloff conditions are compatible with this θ, l and C, and so the presymplectic form is uniquely determined on any Cauchy surface. We won't write it down explicitly now, because we are more interested in what it implies for the Hamiltonian generators of symmetries.

Before getting to that though, there is a subtlety we should discuss. In the above derivation, we keep the Cauchy surfaces Σ_-, Σ_+ fixed as we take the $r \to \infty$ limit. Then, in this limit the boundaries $\partial\Sigma_-, \partial\Sigma_+$ go to spatial infinity in this limit. Thus, the presymplectic form is defined on Cauchy surfaces which extend to spatial infinity. However, in principle there is no reason why we can't move the Cauchy surfaces simultaneously with the $r \to \infty$ limit. In particular, we can consider a limit in which $\partial\Sigma_+$ goes to future null infinity and $\partial\Sigma_-$ goes to past null infinity. By doing this, we can also define the presymplectic form on a Cauchy surface which extends to null infinity.

As we have shown, the presymplectic form is independent of the choice of Cauchy surface when the boundary conditions are satisfied. Therefore, in principle, using a Cauchy surface that extends to spatial infinity should give exactly the same results as a Cauchy surface that extends to null infinity, when we compute the presymplectic form, or any Hamiltonians. However, whether this is actually true in practice depends on the relationship between the boundary conditions at spatial infinity and those at null infinity, and, as we already stated, understanding this relationship fully is a very difficult mathematical problem. There are also possible issues that can arise from the fact that we are taking multiple limits simultabeously – in general, strange things can happen when we mix limits in this way.

There are a couple of reasons to use a Cauchy surface that has its boundary at null infinity. The first reason is that the falloff conditions are better understood near null infinity. The second is that the symmetries of the theory become easier to analyse at null infinity. For these reasons, we will use a Cauchy surface with its boundary at null infinity, and assume it is valid to do so.

(Note however that we could also do the following analysis at spatial infinity.)

10.4 Lagrangian symmetries

There are two types of gauge transformation in Einstein-Maxwell theory:

- Electromagnetic gauge transformations $A_\mu \to A_\mu + \partial_\mu\lambda$.
- Diffeomorphisms $g_{\mu\nu} \to g_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu}$, $A_\mu \to A_\mu + \mathcal{L}_\xi A_\mu$.

For these gauge transformations to become proper Lagrangian symmetries, they must respect the boundary conditions, or in this case the falloff conditions. Let us see explicitly how this works for the electromagnetic gauge transformation at future null infinity. The falloff conditions are

$$A_r = 0, \quad A_u = \mathcal{O}\Big(\frac{1}{r}\Big), \quad A_z = A_{\bar{z}} = \mathcal{O}(1), \tag{10.24}$$

so we must also have

$$\partial_r\lambda = 0, \quad \partial_u\lambda = \mathcal{O}\Big(\frac{1}{r}\Big), \quad \partial_z\lambda = \partial_{\bar{z}}\lambda = \mathcal{O}(1). \tag{10.25}$$

The solution to these conditions is

$$\lambda(u, r, z, \bar{z}) = \mu(z, \bar{z}) + \mathcal{O}\Big(\frac{1}{r}\Big) \tag{10.26}$$

for some function μ of $z, \bar{z}$. Thus the most general allowed Maxwell gauge transformation asymptotes to a function on the 2-sphere.

We can do a similar albeit much lengthier computation to find the most general allowed vector field ξ for a boundary condition repecting diffeomorphism. One finds that such ξ has the following form near future null infinity:

$$\xi^u = f + \frac{1}{2}u\mathrm{D}_A Y^A, \tag{10.27}$$

$$\xi^r = -\frac{1}{2}r\mathrm{D}_A Y^A + \frac{1}{2}\mathrm{D}^2\xi^u + \mathcal{O}\Big(\frac{1}{r}\Big), \tag{10.28}$$

$$\xi^A = Y^A - \frac{1}{r}\mathrm{D}^A\xi^u + \mathcal{O}\Big(\frac{1}{r^2}\Big), \tag{10.29}$$

for some functions $f, Y^z, Y^{\bar{z}}$ of $z, \bar{z}$, where Y^A obeys the conformal Killing equation on the 2-sphere

$$\mathrm{D}_A Y_B + \mathrm{D}_B Y_A = \frac{1}{2}\gamma_{AB}\mathrm{D}_C Y^C. \tag{10.30}$$

The diffeomorphisms that these ξ generate are known as BMS symmetries, and they generate an algebra called the BMS algebra. The BMS group contains the Poincaré algebra as a subalgebra – we might have expected this from the fact that we are in an asymptotically flat setting, and the Poincaré group is the symmetry group of flat spacetime.

- For $f = 1$ and $Y = 0$, we get Poincaré time translation.
- For f an $l = 1$ spherical harmonic, and $Y = 0$, we get Poincaré spatial translations.
- For $f = 0$, and Y^A a global conformal Killing vector on the 2-sphere, we get Poincaré boosts and rotations.

But it contains much more than just the Poincaré algebra. The transformation for a general f is known as a supertranslation, and the transformation for a general Y^A is known as a superrotation. There are infinitely many of them, because there are infinitely many choices of f and Y.

10.5 Generators of symmetries

Now we are in a position to write down the generators of these symmetries. The presymplectic potential θ, and the boundary object C, are in the same form as we have previously considered, so we can use the formulae we have previously derived.

In particular, for a combined diffeomorphism and Maxwell gauge transformation

$$A \to A + \mathcal{L}_\xi A + \mathrm{d}\lambda\,, \qquad g_{\mu\nu} \to g_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu}, \tag{10.31}$$

The Hamiltonian generator is the sum of the expressions we found in lectures 7 and 8:

$$H = \frac{1}{8\pi G}\int_{\partial\Sigma} \tau^\mu \xi^\nu \Big(K_{\mu\nu} - h_{\mu\nu}K\Big)\varepsilon_{\partial\Sigma} + \frac{1}{e^2}\int_{\partial\Sigma} \lambda * F. \tag{10.32}$$

At this stage we can substitute in the explicit form of the metric, gauge potential, diffeomorphism vector and Maxwell gauge parameter, taking into account all of their various falloffs. Without going into detail, one finds:

$$H = \int_{\partial\Sigma} \mathrm{d}^2 z\, \sqrt{\gamma}\Big[\frac{m_B}{4\pi G} f + \Big(\frac{N_A}{8\pi G} + \frac{\mathcal{E}A_A}{e^2}\Big)Y^A + \frac{\mathcal{E}}{e^2}\mu\Big], \tag{10.33}$$

where $\mathcal{E}$ is defined by $A_u = \frac{\mathcal{E}}{r} + \mathcal{O}(1/r^2)$.

Now we can finally get expressions for the energy, angular momentum and electric charge.

- The energy is the generator of Poincaré time translations, so to get it we set $f = 1, Y = 0, \mu = 0$:
$$E = \int_{\partial\Sigma} \mathrm{d}^2 z\, \sqrt{\gamma}\frac{m_B}{4\pi G}. \tag{10.34}$$
- The angular momentum is the generator of rotation around a particular axis, so to get we set $f = 0, \mu = 0$ and pick a globally defined Y:
$$J = \int_{\partial\Sigma} \mathrm{d}^2 z\, \sqrt{\gamma}\Big(\frac{N_A}{8\pi G} + \frac{\mathcal{E}A_A}{e^2}\Big)Y^A. \tag{10.35}$$
Note that both the gravitational field (through the angular momentum aspect N_A) and the electromagnetic field (through the combination $\mathcal{E}A_A$) contribute to the total angular momentum.

- The electric charge is the generator of a constant Maxwell gauge transformation, so we set $f = Y = 0$, $\mu = 1$:
$$q = \int_{\partial\Sigma} \mathrm{d}^2 z \, \sqrt{\gamma} \frac{\mathcal{E}}{e^2}. \tag{10.36}$$

10.6 Black holes

Lets now compute these quantities for the most general stationary asymptotically flat black hole, described by the Kerr-Newman solution. In Bondi coordinates, and in units where $G = e = 1$, the metric takes the form

$$\begin{aligned} \mathrm{d}s^2 = &-\left(1 - \frac{2Mr - Q^2}{R^2}\right) \mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}r - 2\frac{a\sin^2\theta}{R^2}(2Mr - Q^2)\,\mathrm{d}u\,\mathrm{d}\phi \\ &+ 2a\sin^2\theta\,\mathrm{d}r\,\mathrm{d}\phi + R^2\,\mathrm{d}\theta^2 - \frac{\sin^2\theta}{R^2}\Big(\Delta a^2 \sin^2\theta - (a^2 + r^2)^2\Big)\,\mathrm{d}\phi^2\,, \end{aligned} \tag{10.37}$$

where M, Q, a are constant parameters, and

$$R^2 = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 + a^2 - 2Mr + Q^2. \tag{10.38}$$

From these, we can take r large and read off the mass and angular momentum aspects:

$$m_B = M, \qquad N_\phi = 3Ma\sin^2\theta, \qquad N_\theta = 0. \tag{10.39}$$

Furthermore, one can show that

$$\mathcal{E} = Q, \qquad A_A = \mathcal{O}\left(\frac{1}{r}\right). \tag{10.40}$$

The axis of symmetry rotates around ϕ, so the appropriate 2-sphere vector to find the total angular momentum is $Y = \frac{\partial}{\partial\phi}$.

Using these, one finds the following:

$$E = \frac{M}{4\pi} \underbrace{\int_{\partial\Sigma} \mathrm{d}^2 z \, \sqrt{\gamma}}_{=\text{ volume of 2-sphere }=4\pi} = M, \tag{10.41}$$

$$J = \frac{3Ma}{8\pi} \underbrace{\int_{\partial\Sigma} \mathrm{d}^2 z \, \sqrt{\gamma} \sin^2\theta}_{=8\pi/3} = Ma, \tag{10.42}$$

$$q = Q \underbrace{\int_{\partial\Sigma} \mathrm{d}^2 z \, \sqrt{\gamma}}_{=\text{ volume of 2-sphere }=4\pi} = 4\pi Q. \tag{10.43}$$

In this way we have obtained the energy, angular momentum and electric charge of a general black hole, starting from their definitions as the generators of symmetries.

10.7 Other symmetries

But as we pointed out these are only some of the symmetries. It is natural to ask: what is the physical meaning of the generators of the other symmetries? This is up for debate.

Classically, a set of theorems, called the no-hair theorems, say that a stationary black hole is fully determined by its energy, angular momentum and electric charge. So in the classical case the extra generators would not describe any independent degrees of freedom.

On the other hand, the symmetries could be meaningful quantum mechanically, and lead to new degrees of freedom in black hole spacetimes. This is known as the soft hair proposal – we won't discuss it further.

However, next time, we'll see how the generator of a particular combination of symmetries can be interpreted as the entropy of a black hole. This will provide our first insight into the thermodynamical nature of black holes.