1 Introduction

General covariance is one of the fundamental principles underlying gravity. It says that the laws of physics do not depend on our choice of coordinates. Geometrically speaking, this means that we can apply any diffeomorphism to a physical system without changing its properties. In other words, diffeomorphisms are a kind of gauge symmetry. In this course, we will explore an elegant modern perspective on general covariance, using an approach known as the *covariant phase space* formalism. As the name itself suggests, this formalism tells us how to treat covariant theories using classical Hamiltonian mechanics. Besides of its foundational relevance, over the past decades, this formalism has acquired (and still has) a key role in studying semi-classical aspects of gravitational physics finding several applications and leading to new insights within some of the major approaches to quantum gravity. Indeed, despite of personal tastes or preferences, quantum gravity in its basic essence can be arguably thought of as a quantisation of a classical gravity theory, at least in certain regimes. As such it must inevitably deal with the spacetime diffeomorphism symmetry of the latter. This means that we can learn a lot about some aspects of the quantum theory using the covariant phase space approach. For example, the laws of black hole thermodynamics are a reflection of the fact that black holes are ultimately quantum objects. We will show how key fundamental thermodynamical properties of the black hole, such as its entropy, can be understood using the covariant phase space. We will also discuss more general properties of black hole thermodynamics. Finally, we will explore the connection between gauge symmetry and quantum entanglement, and how this relates to the thermodynamics of spacetime itself.

Structure of the Course (arranged by class session and its topic)

I - Preliminaries

1. Hamiltonian mechanics
2. Geometry of phase space, the symplectic form
3. Gauge symmetry and constraints in mechanics

II - Covariant phase space

4. Covariant field theories
5. Geometry of field space
6. Gauge symmetry and constraints in field theory
7. Global symmetries and large gauge symmetries

III - Application to general relativity and black holes

8. Conserved charges in general relativity
9. Black hole spacetimes and symmetries

10. Energy, angular momentum and electric charge
11. Black hole entropy as a Noether charge
12. The laws of black hole thermodynamics

IV - Advanced topics

13. Entanglement and gauge symmetry
14. The first law of entanglement entropy
15. Spacetime thermodynamics and the Einstein equations

References: The main references on which the notes are based as well as further recommended readings for individual topics will be provided along the flow of the lectures.

Notation and conventions: Comments and remarks to be presented only orally in the class are written in cyan colour.

Fix here common conventions adopted throughout the lecture notes, e.g. Einstein sum convention, index notations, list of symbols...

2 Preliminaries: Constrained Particle Mechanics

Let's fix a common starting point for the main basic ingredients and notions that will turn useful all over the lectures.
The material contained in the 3 sections of this chapter should cover the first 3 lectures.

As it is well know, the modern theoretical description of physical systems from 17th Century Newtonian mechanics, passing through field theories, up to the most recent developments in theoretical and mathematical physics is deeply rooted in two main approaches and their later extensions whose historical developments have benefited from the work of eminent physicists and mathematicians like Euler, Laplace, Lagrange, Legendre, Gauss, Liouville, Poisson, Hamilton, and many others. These are known respectively as the Lagrangian and Hamiltonian formalisms. Despite of putting the emphasis on different aspects, namely

- **Lagrangian formalism:** Simpler to set up (no Poisson brackets, no interpretation of momenta), manifest symmetry content of the theory, starting point for path integral quantisation
- **Hamiltonian formalism:** canonically equipped with the main structures/notions of symplectic geometry, starting point of Dirac constraint algorithm (important for gauge theories), starting point of canonical quantisation (emphasis on Hilbert space and states)

both formalisms share a beautiful structural essence of geometric nature[1]. In these lectures, we will mainly be focusing on the Hamiltonian formalism and the inbuilt symplectic geometry of canonical phase space formulation, the latter being a prerequisite to enter the details of the covariant phase space approach to field theories.

To illustrate the main notions and geometrical tools in a simple setup before moving to the infinite-dimensional realm of field theories we will now:

- consider finite-dimensional mechanical systems
- recast known concepts and results in intrinsic geometric terms
- give simple illustrative examples, focussing on the concept of covariance

[1] Even though the phase space of Hamiltonian formalism, often identified with the cotangent bundle $T^*\mathcal{Q}$ over the configuration space manifold $\mathcal{Q}$, comes to be canonically equipped with a symplectic structure, it is possible to phrase the Lagrangian formalism on the tangent bundle $T\mathcal{Q}$ in terms of symplectic geometry as well (see e.g. [4] and references therein). The resulting symplectic structure however depends on the given Lagrangian function or in other words, we need some amount of dynamical information to define it. As we will discuss soon, on $T^*\mathcal{Q}$ instead there is a naturally and globally defined symplectic form which has a purely geometric nature since its definition relies only on the structure of the cotangent bundle with no need of additional informations. Such a symplectic/canonical formalism on $T\mathcal{Q}$ has been then used also as starting point for a Lagrangian counterpart of Dirac's theory of constrained systems [5, 6].

2.1 Further Reading

- Symplectic mechanics: [1–4]
- Theory of constraints: [5, 8–11]
- Reparametrisation invariant systems: [14]

2.2 Unconstrained Mechanics

Let us consider a mechanical system described by $N < \infty$ configuration variables q^i (e.g. positions) and velocities $\dot{q}^i$, $i = 1, \dots, N$, whose dynamics is governed by the action functional

$$S[q^i, \dot{q}^i] = \int \mathrm{d}t\, L(q^i, \dot{q}^i) \ , \tag{2.1}$$

with time-independent (first-order) Lagrangian[2]

$$L(q^i, \dot{q}^i) \equiv L(q^1, \dots, q^N, \dot{q}^1, \dots, \dot{q}^N) \ . \tag{2.2}$$

Here dots denote derivatives w.r.t. "time", the latter being the parameter of the evolution which enters the action functional and is not necessarily a coordinate time.

The generalised positions q^i and the associated velocities $\dot{q}^i$ play the role of local coordinates on the tangent bundle $T\mathcal{Q}$ over the N-dimensional configuration manifold $\mathcal{Q}$ of the system. The Lagrangian comes then to be identified with a real-valued smooth function on the tangent bundle, i.e. $L \in \mathcal{F}(T\mathcal{Q})$.
Least action principle $\delta S = 0 \ \forall \ \delta q^i$ with boundary conditions $\delta q^i|_{t_{\text{in}}} = \delta q^i|_{t_{\text{fin}}} = 0$ yields the Euler-Lagrange (EL) EOMs

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0 \qquad \Leftrightarrow \qquad \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\,\ddot{q}^j = \frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\,\dot{q}^j \tag{2.3}$$

which are a system of N second-order ordinary differential equations[3].
In particular – and this will be crucial in the following discussion as well as a key difference w.r.t. the case of constrained systems with gauge symmetry – we see that if the the following regularity condition for the Lagrangian

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right) \neq 0 \qquad \textbf{regular Lagrangian} \tag{2.4}$$

holds true, then evolution is unique in the sense that all $\ddot{q}^i$ can be completely determined as functions of q^i and $\dot{q}^i$.

[2] We refer to [7] and references therein for the analysis of higher-order theories (with constraints).

[3] Note that, even though Eqs. (2.3) might look as being partial differential equations at first sight, they are classified as ordinary differential equations at least for the case of mechanical systems (not field theory) where, given a Lagrangian, the only variable w.r.t. the variational principle is defined is t and $q^i(t), \dot{q}^i(t)$ are the unknown functions of it to be determined. This is in fact compatible with the observation of the EL equation being just a rewriting of Newtonian EoM lifted to $T\mathcal{Q}$. In the context of the *inverse problem of the calculus of variation*, on the contrary, one starts with the EOM and regards (2.3) as a system of partial differential equations for the unknown Lagrangian functions.

Intrinsic geometric formulation of Euler-Lagrange equations

The EL equations (2.3) can be written in differential-geometric terms as:

$$\mathcal{L}_\Gamma \theta_L - \mathrm{d}L = 0 \qquad \text{or equivalently} \qquad \iota_\Gamma \omega_L = \mathrm{d}\mathcal{E}_L \tag{2.5}$$

where

- d, $\mathcal{L}_X$, and ι_X $(X \in \mathfrak{X}(T\mathcal{Q}))$ denote exterior derivative, Lie derivative, and contraction
  other common notations for $\iota_X\omega$ *are* $X \lrcorner\, \omega$, $X \cdot \omega$, $\omega(X, \cdot)$
- $\Gamma \in \mathfrak{X}(T\mathcal{Q})$ the tangent vector field to the curve $(q^i(t), \dot{q}^i(t))$ in $T\mathcal{Q}$ whose coordinate expression is given by
  $$\Gamma = \dot{q}^i \frac{\partial}{\partial q^i} + \ddot{q}^i \frac{\partial}{\partial \dot{q}^i}$$
- $\theta_L := \frac{\partial L}{\partial \dot{q}^i}\mathrm{d}q^i \in \Omega^1(T\mathcal{Q})$ and $\omega_L := -\mathrm{d}\theta_L \in \Omega^2(T\mathcal{Q})$ closed 2-form which is non-degenerate (hence symplectic) *iff* $L \in \mathcal{F}(T\mathcal{Q})$ is regular
- energy associated with the Lagrangian: $\quad \mathcal{E}_L := \iota_\Gamma \theta_L - L$

Later in the notes, when discussing field theories, it will be convenient to distinguish between exterior derivatives in spacetime (d) and in field space (δ), which are in fact different spaces with their own differential calculus. However, now we use d to denote differentiation w.r.t. both t and $q, \dot{q}$ without making any distinction between the exterior derivative w.r.t. dynamical variables and time derivatives. Thinking about particle mechanics where the configuration variables q^i are often identified with the positions of the particles (e.g. word-line points which lie in spacetime), the reason for such a simplification should be clear.

Proof. Let us start by showing how (2.3) in the $(q^i, \dot{q}^i)$ local coordinates can be recast into the first expression in (2.5). To this aim, let us notice that the l.h.s. of the EL equations (2.3) transforms as the set of components of a covector under point transformations $(q^i, \dot{q}^i) \mapsto (Q^i(q), \dot{Q}^i(q, \dot{q}))$ on $T\mathcal{Q}$. Indeed, first of all we note that $q^k = q^k(Q(t), t)$ has both an explicit t-dependence and an implicit dependence through Q. Hence, we have

$$\dot{q}^k = \frac{\mathrm{d}}{\mathrm{d}t} q^k(Q(t), t) = \frac{\partial q^k}{\partial Q^i}\dot{Q}^i + \frac{\partial q^k}{\partial t} \ , \tag{2.6}$$

from which it follows that

$$\frac{\partial \dot{q}^k}{\partial \dot{Q}^i} = \frac{\partial q^k}{\partial Q^i} \ , \tag{2.7}$$

$$\frac{\partial \dot{q}^k}{\partial Q^i} = \frac{\partial^2 q^k}{\partial Q^i \partial Q^j}\dot{Q}^i + \frac{\partial^2 q^k}{\partial Q^i \partial t} = \frac{\partial}{\partial Q^i}\left(\frac{\partial q^k}{\partial Q^j}\right)\dot{Q}^i + \frac{\partial}{\partial t}\left(\frac{\partial q^k}{\partial Q^j}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial q^k}{\partial Q^j}\right) \ , \tag{2.8}$$

Under the above transformation we also have $L \mapsto \tilde{L}(Q,\dot{Q}) = L(q(Q),\dot{q}(q,\dot{Q}))$, so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \dot{Q}^i} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^k}\frac{\partial \dot{q}^k}{\partial \dot{Q}^i}\right) \overset{(2.7)}{=} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^k}\frac{\partial q^k}{\partial Q^i}\right) \overset{(2.8)}{=} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^k}\right)\frac{\partial q^k}{\partial Q^i} + \frac{\partial L}{\partial \dot{q}^k}\frac{\partial \dot{q}^k}{\partial q^j}\frac{\partial q^j}{\partial Q^i}\ ,$$

and

$$\frac{\partial \tilde{L}}{\partial Q^i} = \frac{\partial L}{\partial q^k}\frac{\partial q^k}{\partial Q^i} + \frac{\partial L}{\partial \dot{q}^k}\frac{\partial \dot{q}^k}{\partial q^j}\frac{\partial q^j}{\partial Q^i}\ ,$$

Therefore, the last terms on the r.h.s. of the above equations compensate each other and we get the desired covector-like transformation behaviour, namely

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \dot{Q}^i} - \frac{\partial \tilde{L}}{\partial Q^i} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k}\right)\frac{\partial q^k}{\partial Q^i}\ .$$

Multiplying then both sides of the first equation in (2.3) by $\mathrm{d}q^i$, identifying the total time derivative $\frac{\mathrm{d}}{\mathrm{d}t}$ along the trajectories $(q^i(t),\dot{q}^i(t))$ with $\mathcal{L}_\Gamma$, and recalling that the $\mathrm{d}q^i$s are a basis of independent 1-forms, we can rewrite the EL equations (2.3) as

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i}\right)\mathrm{d}q^i &= \mathcal{L}_\Gamma\left(\frac{\partial L}{\partial \dot{q}^i}\mathrm{d}q^i\right) - \frac{\partial L}{\partial \dot{q}^i}\mathcal{L}_\Gamma(\mathrm{d}q^i) - \frac{\partial L}{\partial q^i}\mathrm{d}q^i \\ &= \mathcal{L}_\Gamma\left(\frac{\partial L}{\partial \dot{q}^i}\mathrm{d}q^i\right) - \frac{\partial L}{\partial \dot{q}^i}\mathrm{d}(\mathcal{L}_\Gamma q^i) - \frac{\partial L}{\partial q^i}\mathrm{d}q^i \\ &= \mathcal{L}_\Gamma\theta_L - \mathrm{d}L \qquad \text{with} \qquad \theta_L := \frac{\partial L}{\partial \dot{q}^i}\mathrm{d}q^i\ .\end{aligned}$$

Using now the definition of θ_L[4], the local coordinate expression for $\omega_L = -\mathrm{d}\theta_L$ reads as

$$\begin{aligned}\omega_L &= -\mathrm{d}\left(\frac{\partial L}{\partial \dot{q}^i}\mathrm{d}q^i\right)\wedge \mathrm{d}q^i \\ &= -\left(\frac{\partial^2 L}{\partial \dot{q}^i\partial \dot{q}^j}\mathrm{d}\dot{q}^j\wedge \mathrm{d}q^i + \frac{\partial^2 L}{\partial \dot{q}^i\partial q^j}\mathrm{d}q^j\wedge \mathrm{d}q^i\right) \\ &= \frac{\partial^2 L}{\partial \dot{q}^i\partial \dot{q}^j}\mathrm{d}q^i\wedge \mathrm{d}\dot{q}^j + \frac{1}{2}\left(\frac{\partial^2 L}{\partial \dot{q}^i\partial q^j} - \frac{\partial^2 L}{\partial \dot{q}^j\partial q^i}\right)\mathrm{d}q^i\wedge \mathrm{d}q^j\ .\end{aligned}$$

The 2-form ω_L is obviously closed ($\mathrm{d}\omega_L = 0$) as $\mathrm{d}^2 = 0$. Moreover, it is non-degenerate *iff* L is regular. In fact, for any vector field $X = f^i\frac{\partial}{\partial q^i} + g^i\frac{\partial}{\partial \dot{q}^i} \in \mathfrak{X}(T\mathcal{Q})$, we have

$$\iota_X\omega_L = \frac{\partial^2 L}{\partial \dot{q}^i\partial \dot{q}^j}f^j\mathrm{d}q^i + \left(\frac{\partial^2 L}{\partial \dot{q}^i\partial \dot{q}^j}g^j + \left(\frac{\partial^2 L}{\partial \dot{q}^i\partial q^j} - \frac{\partial^2 L}{\partial \dot{q}^j\partial q^i}\right)f^j\right)\mathrm{d}q^i\ ,$$

from which we see that the non-degeneracy condition for ω_L, i.e.

$$\iota_X\omega_L = 0 \qquad \Leftrightarrow \qquad X = 0\ \ (f^i, g^i = 0\ \forall i)$$

[4] Note that θ_L is related to the boundary term in the variation of the action (2.1) as

$$\delta S = \int \mathrm{d}t\left(\frac{\partial L}{\partial q^i}\mathrm{d}q^i + \frac{\partial L}{\partial \dot{q}^i}\mathrm{d}\dot{q}^i\right) = \int \mathrm{d}t\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i}\right)\mathrm{d}q^i\right] + \left(\frac{\partial L}{\partial \dot{q}^i}\mathrm{d}q^i\right)\Bigg|_{t_i}^{t_f}\ .$$

We will return to this important relation between the variation of the Lagrangian, the EOM, and the Cartan 1-form for field theories in chapter ?? below.

is equivalent to the condition

$$\det \left\| \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right\| \neq 0 \, ,$$

in which case the above equation in local coordinates can be inverted (with the l.h.s. equal to zero in virtue of the non-degeneracy of ω_L) w.r.t. f^j or g^j. Thus, for regular Lagrangians, ω_L is a closed non-degenerate 2-form on $T\mathcal{Q}$ and as such it is called a *Lagrangian symplectic structure*. Finally, using Cartan's identity $\mathcal{L}_\Gamma = \iota_\Gamma \mathrm{d} + \mathrm{d}\iota_\Gamma$, we can write

$$0 = \mathcal{L}_\Gamma \theta_L - \mathrm{d}L = -\iota_\Gamma \omega_L + \mathrm{d}(\iota_\Gamma \theta_L - L) \qquad \text{i.e.} \qquad \iota_\Gamma \omega_L = \mathrm{d}\mathcal{E}_L \tag{2.9}$$

which is the second version of (2.5) with *Lagrangian energy*[5] $\mathcal{E}_L = \iota_\Gamma \theta_L - L$. □

The advantage of such a geometric rewriting in terms of differential forms and vector fields resides into the intrinsic character of Cartan calculus which does not depends on the specific choice of local coordinate charts. Moreover, as we will discuss in the coming subsection, the symplectic version of EL equations has a straightforward counterpart in the canonical Hamiltonian formalism.

2.2.1 Legendre Transform, Phase Space, and Symplectic Structure

As discussed previously under Eq. (2.3), in the case of a regular Lagrangian (cfr. Eq. (2.4)), the evolution is unique and $\ddot{q}^i$ can be completely determined as functions of q^i and $\dot{q}^i$. Equivalently, velocities $\dot{q}^i$ can be completely determined as functions of generalized coordinates q^i and momenta $p_i = \frac{\partial L}{\partial \dot{q}^i}$ for regular Lagrangians. In this case then, the Legendre transformation[6]

$$FL : \, T\mathcal{Q} \longrightarrow T^*\mathcal{Q} \qquad \text{by} \qquad (q^i, \dot{q}^i) \longmapsto \left(q^i, p_i = \frac{\partial L}{\partial \dot{q}^i} \right) \, , \tag{2.10}$$

provides us with a one-to-one correspondence between the Lagrangian and the Hamiltonian descriptions. In the Hamiltonian description, (q^i, p_i), $i = 1, \ldots, N$, are local coordinates on the cotangent bundle $T^*\mathcal{Q}$ over the configuration manifold $\mathcal{Q}$. Dynamics on $T\mathcal{Q}$ described by the EL equations can be translated on $T^*\mathcal{Q}$ as follows:

[5] To get a glimpse of such a name, consider for instance a single particle Lagrangian of the kind $L = \frac{1}{2}\delta_{ij}\dot{q}^i\dot{q}^j - U(q)$. In this case, $\iota_\Gamma \theta_L = \delta_{ij}\dot{q}^i\dot{q}^j$ and hence $\mathcal{E}_L = \iota_\Gamma \theta_L - L = \frac{1}{2}\delta_{ij}\dot{q}^i\dot{q}^j + U(q) = T + U$. More generally, it is possible to recast the local coordinate expression for the EL equation in terms of analogous equations to Hamilton equations with $\mathcal{E}_L$ playing a role analogous to the Hamiltonian.

[6] The notation FL as often denoted in the mathematical physics literature comes from the fact that, geometrically speaking, the Legendre transform is identified with the fiber derivative of the Lagrangian, or more precisely of the mapping $\mathsf{L} : T\mathcal{Q} \to \mathcal{Q} \times \mathbb{R}$ by $(q, \dot{q}) \mapsto (q, L(q, \dot{q}))$ between bundles over $\mathcal{Q}$ with $F\mathsf{L} : T_q\mathcal{Q} \to \mathsf{Lin}(T_q\mathcal{Q}, \mathbb{R}) \equiv T_q^*\mathcal{Q}$.

Hamilton equations (intrinsic formulation)

EL eqs on $T\mathcal{Q}$ $\qquad T\mathcal{Q} \ni (q^i, \dot{q}^i) \xrightarrow{FL} (q^i, p_i) \in T^*\mathcal{Q}$ $\qquad$ Hamilton eqs on $T^*\mathcal{Q}$

$$\iota_\Gamma \omega_L = \mathrm{d}\mathcal{E}_L \qquad\qquad \iota_{X_H}\omega = \mathrm{d}H \tag{2.11}$$

- Cartan 1-form: $\quad \theta \in \Omega^1(T^*\mathcal{Q}) \quad$ s.t. $\quad \theta_L = (FL)^*\theta$

$$\theta = p_i \mathrm{d}q^i \tag{2.12}$$

- symplectic 2-form: $\quad \omega \in \Omega^2(T^*\mathcal{Q}) \quad (\mathrm{d}(FL)^* = (FL)^*\mathrm{d})$

$$\omega = -\mathrm{d}\theta = \mathrm{d}q^i \wedge \mathrm{d}p_i \tag{2.13}$$

- Hamiltonian vector field: $\quad X_H \in \mathfrak{X}(T^*\mathcal{Q}) \quad$ s.t. $\quad X_H = (FL)_*\Gamma$

$$X_H = \dot{q}^i \frac{\partial}{\partial q^i} + \dot{p}_i \frac{\partial}{\partial p_i} = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} \tag{2.14}$$

- Hamiltonian: $\quad H \in \mathcal{F}(T^*\mathcal{Q}) \quad$ s.t. $\quad \mathcal{E}_L = (FL)^*H$

$$H(q,p) = p_i \dot{q}^i(q,p)) - L(q, \dot{q}(q,p)) \tag{2.15}$$

Some references may use a different convention with $\omega = \mathrm{d}\theta = \mathrm{d}p_i \wedge \mathrm{d}q^i$ and $\iota_{X_H}\omega = -\mathrm{d}H$, which simply amounts to a similar redefinition for ω_L and a minus sign appearing on the r.h.s. of the intrinsic form of EL equations.

Proof. For regular Lagrangians, the Legendre map provides us with a diffeomorphism from $T\mathcal{Q}$ to $T^*\mathcal{Q}$. As such it is bijective and we can pushforward covectors (forms) along the map by pulling them back along the inverse map, say $(FL)^* = (FL)_*^{-1}$. Therefore, we have

$$H = (FL)_*\mathcal{E}_L = (FL)_*(\iota_\Gamma\theta_L - L) = (FL)_*\left(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L(q,\dot{q})\right) = p_i\dot{q}^i(q,p) - L(q,\dot{q}(q,p)) \,. \tag{2.16}$$

X_H is the tangent vector field to the trajectories $(q^i(t), p_i(t))$ in $T^*\mathcal{Q}$, i.e. in local coordinates we have

$$X_H = \dot{q}^i \frac{\partial}{\partial q^i} + \dot{p}_i \frac{\partial}{\partial p_i} \,, \tag{2.17}$$

so that the Hamilton EOMs yield algebraic equations for (the components of) X_H, namely

$$\begin{array}{l} \iota_{X_H}\omega = \iota_{X_H}(\mathrm{d}q^i \wedge \mathrm{d}p_i) = \dot{q}^i \mathrm{d}p_i - \dot{p}_i \mathrm{d}q^i \\ \mathrm{d}H = \frac{\partial H}{\partial q^i}\mathrm{d}q^i + \frac{\partial H}{\partial p_i}\mathrm{d}p_i \end{array} \quad \Rightarrow \quad \begin{cases} \dot{q}^i = \frac{\partial H}{\partial p_i} \\ \dot{p}_i = -\frac{\partial H}{\partial q^i} \end{cases} \tag{2.18}$$

which is the familiar local coordinate expression of Hamilton equations.

As before, $\omega = \mathrm{d}\theta$ is closed as $\mathrm{d}^2 = 0$. Moreover, ω_L is non-degenerate (*iff* L is regular) so in this case ω is non-degenerate too. $\square$

The $2N$-dimensional manifold $\mathcal{P} = (T^*\mathcal{Q}, \omega)$ is thus a symplectic manifold and is called the *phase space* of the system[7]. As we will discuss into the next section, this is not the case with gauge symmetry in which case ω becomes degenerate (*pre-symplectic*).

The vector field X_H is called *Hamiltonian vector field* associated to the Hamiltonian function H. More generally, we have the following definitions

Hamiltonian vector field

Let $f \in \mathcal{F}(\mathcal{P})$ be a *sufficiently smooth* phase space function. A vector field $X_f \in \mathfrak{X}(\mathcal{P})$ is said to be the *Hamiltonian vector field* associated to f if

$$\iota_{X_f}\omega = \mathrm{d}f \ , \tag{2.19}$$

and its local coordinate expression is given by

$$X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i} \ . \tag{2.20}$$

Poisson bracket (PB)

Let $f, g \in \mathcal{F}(\mathcal{P})$ and let X_f, X_g be their Hamiltonian vector fields. The map

$$\{\cdot,\cdot\} : \mathcal{F}(\mathcal{P}) \times \mathcal{F}(\mathcal{P}) \longmapsto \mathcal{F}(\mathcal{P}) \quad \text{by} \quad (f,g) \longmapsto \{f,g\} := \omega(X_f, X_g) \tag{2.21}$$

or in local coordinates

$$\{f,g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} \ , \tag{2.22}$$

defines a Poisson bracket on the algebra $\mathcal{F}(\mathcal{P})$, i.e. it satisfies the properties

i) Skew-symmetry: $\{f,g\} = -\{g,f\} \quad \forall\, f,g \in \mathcal{F}(\mathcal{P})$

ii) Bilinearity: $\{\alpha f + \beta g, h\} = \alpha\{f,h\} + \beta\{g,h\} \quad \forall\, f,g,h \in \mathcal{F}(\mathcal{P})\,,\ \alpha,\beta \in \mathbb{R}$

iii) Leibniz rule: $\{f,gh\} = \{f,g\}h + g\{f,h\} \quad \forall\, f,g,h \in \mathcal{F}(\mathcal{P})$

iv) Jacobi identity: $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0 \quad \forall\, f,g,h \in \mathcal{F}(\mathcal{P})$

The set of phase space functions forms an algebra over $\mathbb{R}$ (w.r.t. pointwise addition and multiplication). The Poisson bracket adds the structure of a Poisson algebra.

Proof. i) and ii) follow directly from the above definitions due to skewsymmetry and lin-

[7]The case of odd-dimensional manifolds can be described by the "odd-dimensional cousin" of symplectic geometry known as contact geometry and has been used in recent years to develop a geometric description of dissipative systems as well as thermodynamics, with interesting connections to information geometry.

earity of ω. As for iii), let us note that

$$\begin{aligned}\{f,g\} &= \omega(X_f,X_g) = \iota_{X_g}\iota_{X_f}\omega = \iota_{X_g}\mathrm{d}f = \mathcal{L}_{X_g}f \\ &= -\omega(X_g,X_f) = -\iota_{X_f}\iota_{X_g}\omega = -\iota_{X_f}\mathrm{d}g = -\mathcal{L}_{X_f}g \ ,\end{aligned} \tag{2.23}$$

from which, recalling the coordinate expression (2.20) for Hamiltonian vector fields together with the action of Lie derivatives on functions $\mathcal{L}_{X_g}f = X_g(f)$, it follows that

$$\{f,g\} = \mathcal{L}_{X_g}f = X_g(f) = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} \ . \tag{2.24}$$

Moreover, we also have that $\{f,gh\} = -\mathcal{L}_{X_f}(gh)$ so that iii) follows directly from the Leibniz rule for Lie derivatives acting on functions. The proof of Jacobi identity iv) is a bit more lengthy and we will omit it here for the sake of brevity. It can be also checked by direct computation using the coordinate expressions of Poisson brackets. $\square$

In particular, for $g=H$, we have

$$\mathcal{L}_{X_H}f = \omega(X_f,X_H) = \{f,H\} \ , \tag{2.25}$$

and, since along the integral curves of X_H we have $\mathcal{L}_{X_H} = \frac{\mathrm{d}}{\mathrm{d}t}$, we get

$$\dot{f} = \frac{\mathrm{d}f}{\mathrm{d}t} = \{f,H\} \ , \tag{2.26}$$

i.e., f obeys the canonical equations of motion in PB notation or equivalently, X_H is the dynamic vector field. In other words, H is the generator of time translations and, as schematically depicted in Fig. 1, dynamical evolution is a flow in phase space with trajectories identified with the integral curves of X_H.

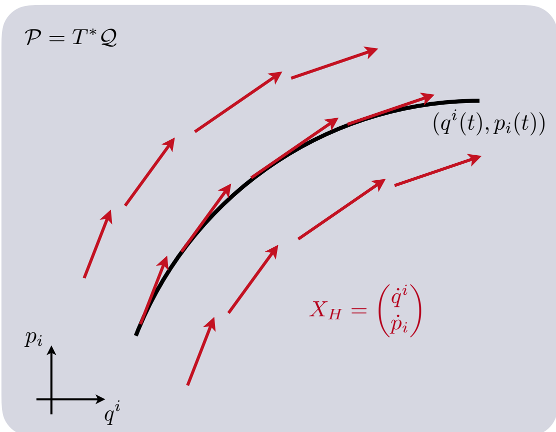

Figure 1. Dynamics as a flow in phase space. Trajectories $(q^i(t), p_i(t))$ are integral curves (black line) of the Hamiltonian vector field X_H tangent to them (red arrows).

Tha Hamiltonian flow equation (2.25) provides us with the infinitesimal version of the time variation of a phase space function. The flow can be then explicitly exponentiated as

$$\begin{aligned} f(q(t),p(t)) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(\frac{\mathrm{d}^n f}{\mathrm{d}t^n}\Big|_{t=0} \right) \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \{f,H\}_{(n)}\big|_{q=q_0,p=p_0} \\ &= e^{t\{\cdot,H\}} f(q,p)\big|_{q=q_0,p=p_0} \\ &= e^{t\,X_H} f(q,p)\big|_{q=q_0,p=p_0} \end{aligned} \tag{2.27}$$

where $\{f,H\}_{(n+1)} := \{\{f,H\}_{(n)},H\}$ with $\{f,H\}_{(0)} := f$. In particular, constants of motion $f \in \mathcal{F}(\mathcal{P})$ s.t. $\dot{f} = \mathcal{L}_{X_H} f = 0$ (preserved along the flow of X_H) have vanishing PBs with the Hamiltonian and vice-versa[8].

Summary: systems without gauge symmetry

- regular Lagrangian, Legendre transform 1-to-1, symplectic structure
- distinct points in phase space correspond to distinct physical situations
- Hamiltonian generates a flow in phase space interpreted as physical evolution

Such a description is however not sufficient as we need to include also gauge systems. In case of constrained systems, i.e. with gauge symmetry, there is a redundancy in the phase space description as distinct points in phase space can correspond to the same physical situation. This in turn amounts to the phase space flow between two physical situations being ambiguous and thus cannot be generated by a unique Hamiltonian.

[8]This, together with the Jacobi identity, implies that the PB of any two constants of the motion is itself a constant of the motion.

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