GR lecture 6

The Riemann curvature tensor

I. CARROLL'S BOOK: SECTIONS 3.6, 3.7

II. THE RIEMANN TENSOR FROM PARALLEL TRANSPORT ALONG A LOOP

Consider an infinitesimal "square" closed loop around a point x. We denote the displacements along the loop's sides as A^μ, B^μ, $-A^\mu$ and $-B^\mu$ respectively. The loop's corners are at $x^\mu - A^\mu/2 - B^\mu/2$, $x^\mu + A^\mu/2 - B^\mu/2$, $x^\mu + A^\mu/2 + B^\mu/2$ and $x^\mu - A^\mu/2 + B^\mu/2$. The midpoints of the loop's sides are at $x^\mu - B^\mu/2$, $x^\mu + A^\mu/2$, $x^\mu + B^\mu/2$ and $x^\mu - A^\mu/2$.

Now, consider parallel transport around this loop, calculated to second order in the infinitesimal vectors A^μ and B^μ. First, we must calculate the parallel transport along a single side to second order. Consider e.g. the first side, along which the displacement vector is A^μ. To first order, the basis transformation matrix for this parallel transport is $\delta^\nu_\mu + A^\rho \Gamma^\nu_{\rho\mu}$, or, in matrix notation for the $\mu\nu$ indices, $1 + A^\rho \Gamma_\rho$. To get the second-order result, we should momentarily stop treating A^μ as infinitesimal, and construct the basis transformation matrix as the product of matrices along $N \to \infty$ smaller intervals A^μ/N:

$$M = \left(1 + \frac{1}{N} A^\rho \Gamma_\rho\right)^N \to e^{A^\rho \Gamma_\rho} = 1 + A^\rho \Gamma_\rho + \frac{1}{2}(A^\rho \Gamma_\rho)^2 + \dots \ . \tag{1}$$

Restoring the indices, and following the appropriate index pattern for the matrix product $(A^\rho \Gamma_\rho)_\mu{}^\lambda (A^\sigma \Gamma_\sigma)_\lambda{}^\nu$, the result to second order reads:

$$M_\mu{}^\nu = \delta^\nu_\mu + A^\rho \Gamma^\nu_{\rho\mu} + \frac{1}{2} A^\rho \Gamma^\lambda_{\rho\mu} A^\sigma \Gamma^\nu_{\sigma\lambda} \ . \tag{2}$$

Now, we should remember that we're talking about one side in a square loop, and the midpoint of this side is not at x^μ, but at $x^\mu - B^\mu/2$. Thus, to be exact to second order, we must replace $\Gamma^\nu_{\mu\rho}$ in the second term with $\Gamma^\nu_{\mu\rho}(x - B/2) = \Gamma^\nu_{\mu\rho}(x) - (1/2)B^\sigma \partial_\sigma \Gamma^\nu_{\mu\rho}$:

$$M_\mu{}^\nu = \delta^\nu_\mu + A^\rho \left(\Gamma^\nu_{\rho\mu} - \frac{1}{2} B^\sigma \partial_\sigma \Gamma^\nu_{\rho\mu} + \frac{1}{2} A^\sigma \Gamma^\lambda_{\rho\mu} \Gamma^\nu_{\sigma\lambda}\right) \ . \tag{3}$$

We can now multiply 4 of these matrices, corresponding to the loop's 4 sides, keeping only terms up to second order in A^μ and B^μ. The linear terms cancel, and so do the quadratic terms proportional to $A^\mu A^\nu$ and $B^\mu B^\nu$. We are left with just a term proportional to the antisymmetric matrix $A^{[\mu}B^{\nu]}$, which defines our infinitesimal loop:

$$M_\mu{}^\nu = \delta_\mu^\nu + R^\nu{}_{\mu\rho\sigma} A^\rho B^\sigma \ , \tag{4}$$

where $R^\mu{}_{\nu\rho\sigma}$ is the Riemann tensor:

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\sigma\nu} - \partial_\sigma \Gamma^\mu_{\rho\nu} + \Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\sigma\nu} - \Gamma^\mu_{\sigma\lambda}\Gamma^\lambda_{\rho\nu} \ . \tag{5}$$

In particular, when acting on a vector v^μ or a covector u_μ, this leads to:

$$v^\mu \ \to \ (M^{-1})_\nu{}^\mu v^\nu = v^\mu - R^\mu{}_{\nu\rho\sigma} A^\rho B^\sigma v^\nu \ ; \tag{6}$$

$$u_\mu \ \to \ M_\mu{}^\nu u_\nu = u_\mu + R^\nu{}_{\mu\rho\sigma} A^\rho B^\sigma u_\nu \ . \tag{7}$$

III. COMMUTATOR OF COVARIANT DERIVATIVES ON ARBITRARY TENSORS

In the absence of torsion, the commutator of covariant derivatives $[\nabla_\mu, \nabla_\nu] = \nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu$ precisely encodes the notion of traveling along a closed loop. Thus, the previous result can be reformulated as the action of this commutator on a vector or a covector:

$$[\nabla_\mu, \nabla_\nu] v^\rho = R^\rho{}_{\sigma\mu\nu} v^\sigma \ ; \quad [\nabla_\mu, \nabla_\nu] u_\rho = -R^\sigma{}_{\rho\mu\nu} u_\sigma \ . \tag{8}$$

From here, we can derive the rule for $[\nabla_\mu, \nabla_\nu]$ acting on an arbitrary tensor:

$$\begin{aligned} [\nabla_\mu, \nabla_\nu] T^{\rho\ldots\sigma}{}_{\kappa\ldots\lambda} &= R^\rho{}_{\alpha\mu\nu} T^{\alpha\ldots\sigma}{}_{\kappa\ldots\lambda} + \ldots + R^\sigma{}_{\alpha\mu\nu} T^{\rho\ldots\alpha}{}_{\kappa\ldots\lambda} \\ &\quad - R^\alpha{}_{\kappa\mu\nu} T^{\rho\ldots\sigma}{}_{\alpha\ldots\lambda} - \ldots - R^\alpha{}_{\lambda\mu\nu} T^{\rho\ldots\sigma}{}_{\kappa\ldots\alpha} \ . \end{aligned} \tag{9}$$

This, like its relatives, can be derived via the Leibniz rule by writing the tensor $T^{\rho\ldots\sigma}{}_{\kappa\ldots\lambda}$ as a product of vectors and covectors. The products of first derivatives will cancel between $\nabla_\mu \nabla_\nu(\ldots)$ and $\nabla_\nu \nabla_\mu(\ldots)$, and only second derivatives acting on the individual vectors and covectors will remain, leading to (9).

IV. JACOBI/BIANCHI IDENTITIES

Any operator ∇_μ, even if non-commutative, automatically satisfies the Jacobi identity:

$$[\nabla_\mu, [\nabla_\nu, \nabla_\rho]] + [\nabla_\nu, [\nabla_\rho, \nabla_\mu]] + [\nabla_\rho, [\nabla_\mu, \nabla_\nu]] = 3 \left[\nabla_{[\mu}, [\nabla_\nu, \nabla_{\rho]}] \right] = 0 \ . \tag{10}$$

Let us apply this to a vector v^σ. For the first term in (10), we get:

$$\begin{aligned}[\nabla_\mu, [\nabla_\nu, \nabla_\rho]]\, v^\sigma &= \nabla_\mu [\nabla_\nu, \nabla_\rho] v^\sigma - [\nabla_\nu, \nabla_\rho](\nabla_\mu v^\sigma) \\ &= \nabla_\mu (R^\sigma{}_{\lambda\nu\rho} v^\lambda) + R^\lambda{}_{\mu\nu\rho} \nabla_\lambda v^\sigma - R^\sigma{}_{\lambda\nu\rho} \nabla_\mu v^\lambda \\ &= v^\lambda \nabla_\mu R^\sigma{}_{\lambda\nu\rho} + R^\lambda{}_{\mu\nu\rho} \nabla_\lambda v^\sigma \ .\end{aligned} \tag{11}$$

Now, antisymmetrizing over $\mu\nu\rho$ (or, equivalently, adding the cyclic permutations of $\mu\nu\rho$), we get:

$$v^\lambda (\nabla_\mu R^\sigma{}_{\lambda\nu\rho} + \nabla_\nu R^\sigma{}_{\lambda\rho\mu} + \nabla_\rho R^\sigma{}_{\lambda\mu\nu}) + 3 R^\lambda{}_{[\mu\nu\rho]} \nabla_\lambda v^\sigma = 0 \ . \tag{12}$$

Demanding that this holds for any vector field v^μ, we get the algebraic and differential Bianchi identities:

$$R^\lambda{}_{[\mu\nu\rho]} = 0 \ ; \quad \nabla_\mu R^\sigma{}_{\lambda\nu\rho} + \nabla_\nu R^\sigma{}_{\lambda\rho\mu} + \nabla_\rho R^\sigma{}_{\lambda\mu\nu} = 0 \ . \tag{13}$$

EXERCISES

Exercise 1. *Derive the expression* (5) *for the Riemann tensor directly from one of the commutators* (8)*.*

Exercise 2. *For the Riemann tensor of the Christoffel connection, derive the index symmetry* $R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}$ *from the commutator* $[\nabla_\mu, \nabla_\nu] g_{\nu\rho}$*.*

Exercise 3. *Find the components of the Riemann tensor for:*

- *A 2d sphere, with metric* $ds^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2)$*.*
- *3d flat space in spherical coordinates, with metric* $ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$*.*

Exercise 4. *Find the components of the Riemann tensor at* $t = 0$ *for the metric:*

$$ds^2 = -dt^2 + (1 + \alpha t^2) dx^2 + (1 + \beta t^2) dy^2 + (1 + \gamma t^2) dz^2 \ , \tag{14}$$

where α, β, γ *are constants.*