SPECIAL RELATIVITY HOMEWORK – SOME GAMES WITH 4-CURRENTS

Exercise 1. *Consider the 4-current of a single charged particle:*

$$j^\mu(x^\mu) = q \int_\gamma dy^\mu \, \delta^4(x^\mu - y^\mu) \ , \tag{1}$$

where γ is the particle's worldline $y^\mu(\lambda)$. Show that this current is conserved, i.e. $\partial_\mu j^\mu = 0$.

Exercise 2. *Consider the probability 4-current $J^\mu(x^\mu)$ associated with a complex scalar wavefunction $\psi(x^\mu)$:*

$$J^\mu = \frac{1}{2i}\left(\bar\psi \partial^\mu \psi - \psi \partial^\mu \bar\psi\right) \ . \tag{2}$$

Express J^μ in terms of the wavefunction's absolute value $\rho(x)$ and phase $\theta(x)$:

$$\psi(x) = \rho(x) e^{i\theta(x)} \ . \tag{3}$$

Describe the result's physical meaning.

Exercise 3. *Consider the Fourier decomposition of a free complex scalar field:*

$$\phi(x^\mu) = \int (d^3k)_{nice} \left(a(k) e^{ik_\mu x^\mu} + b^\dagger(k) e^{-ik_\mu x^\mu}\right) \ , \tag{4}$$

where the Lorentz-invariant measure $(d^3k)_{nice}$ over the "mass shell" $k_\mu k^\mu = -m^2$ is:

$$(d^3k)_{nice} = d^4k \, \delta(k_\mu k^\mu + m^2) = \frac{d^3\mathbf{k}}{2\sqrt{\mathbf{k}^2 + m^2}} \ . \tag{5}$$

Evaluate the 4-current (2), with the field (4) in place of the wavefunction $\psi(x)$. Find the total charge in space:

$$Q = \int d^3\mathbf{x} \, J^t(t, \mathbf{x}) \ . \tag{6}$$

Does the charge depend on t? What is the physical meaning of the expression that you found? If you wish, you may ignore the commutators between $a, a^\dagger$ and $b, b^\dagger$.