GENERAL RELATIVITY HOMEWORK – WEEK 8

Exercise 1. *Consider the Schwarzschild metric induced by a central mass M:*

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - 2GM/r} + r^2 d\Omega^2 \ . \qquad (1)$$

Within this spacetime, consider a particle hanging at a constant position (r, θ, ϕ). Note that it must be using some non-gravitational force to keep it from falling inwards. Find the particle's 4-acceleration $\alpha^\mu = Du^\mu/d\tau$ and its magnitude $\sqrt{\alpha_\mu \alpha^\mu}$, as a function of r. What happens as we approach $r = 2GM$?

Exercise 2. *Now, consider a free-falling particle in the Schwarzschild metric, orbiting at a constant radius r. Without loss of generality, assume that the orbit is in the $\theta = \pi/2$ plane. Find the orbit's frequency $\omega \equiv d\phi/dt$. Hint: use the vanishing of α^r to relate u^t and u^ϕ. What is the smallest value of r for which circular orbits are possible? Hint: the particle's worldline must be timelike!*

Exercise 3. *Repeat our derivation of the Schwarzschild metric, but for 3d spacetime (t, r, ϕ) instead of 4d, and with circular instead of spherical symmetry. If you are weak, assume from the start that the metric is time-independent. As we did in class, find the solution to the vacuum equation $R_{\mu\nu} = 0$. Can you make this solution consistent with the presence of a point mass $T^{tt} = M\delta^2(\vec{r})$ at the origin?*