GENERAL RELATIVITY HOMEWORK – WEEK 7

Exercise 1. *Consider the following three metrics:*

Flat plane:	$ds^2 = dr^2 + r^2 d\phi^2$;	(1)
Sphere of radius R:	$ds^2 = R^2(d\theta^2 + \sin^2\theta\, d\phi^2)$;	(2)
Hyperbolic plane of curvature radius R:	$ds^2 = R^2(d\theta^2 + \sinh^2\theta\, d\phi^2)$.	(3)

1. *For each of these, write down all components of the metric g_{ij}, the inverse metric g^{ij} and the Christoffel connection Γ^i_{jk}.*

2. *For the sphere and the hyperbolic plane, find a coordinate transformation $\theta \to r(\theta)$ that makes the $g_{\phi\phi}$ component identical to that of the flat plane. What is g_{rr} in these new coordinates?*

Exercise 2. *Recall the Rindler coordinates for $1+1$-dimensional flat spacetime:*

$$ds^2 = d\rho^2 - \rho^2 d\theta^2 \ . \tag{4}$$

Consider a $\rho = const$ worldline in these coordinates.

1. *Find a proper time parameter τ along this worldline.*

2. *What are the components (u^ρ, u^θ) of the 4-velocity u^μ?*

3. *Find the 4-acceleration $\alpha^\mu = \frac{du^\mu}{d\tau} + \Gamma^\mu_{\nu\rho} u^\nu u^\rho$, and its magnitude $|\alpha| \equiv \sqrt{g_{\mu\nu}\alpha^\mu\alpha^\nu}$.*

Exercise 3. *On the unit sphere $ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$, consider a vector v^i with components (v^θ, v^ϕ). Let us parallel-transport this vector along the circle C of constant θ and varying ϕ:*

$$\frac{dv^i}{d\phi} + \Gamma^i_{\phi j} v^j = 0 \ . \tag{5}$$

1. *Solve this equation for $v^i(\phi)$. Hint: try solving in terms of v^θ and $\hat{v}^\phi \equiv v^\phi \sin\theta$. What is the meaning of this $\hat{v}^\phi$?*

2. *Upon being transported around the full circle C, by what angle does v^i rotate?*

3. *For small θ, show that the answer is proportional to the area inside C.*

4. *What happens at $\theta = \pi/2$?*

5. *What happens if we transport along a circle of constant ϕ instead?*