GENERAL RELATIVITY HOMEWORK – WEEK 7

In this exercise set, we explore various GR equations in the non-relativistic limit of small velocities $\mathbf{v}$ and small deviations $g_{\mu\nu}(x) - \eta_{\mu\nu}$ from the Minkowski metric. You may assume that $g_{\mu\nu} - \eta_{\mu\nu}$ is of order $\mathbf{v}^2$, and ignore any higher orders, such as products of $\mathbf{v}$ and $g_{\mu\nu} - \eta_{\mu\nu}$.

Exercise 1. *Consider the action for a relativistic particle in a curved spacetime:*

$$S = -m \int \sqrt{-g_{\mu\nu}(x) dx^\mu dx^\nu} \ . \tag{1}$$

In the non-relativistic limit, show that this reproduces the action of a non-relativistic particle in a Newtonian gravitational field. Express the Newtonian gravitational potential $\varphi(x)$ in terms of the metric $g_{\mu\nu}(x)$.

Exercise 2. *Consider the geodesic equation for a worldline $x^\mu(\tau)$ parameterized by proper time τ:*

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}(x) \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0 \ . \tag{2}$$

In the non-relativistic limit, notice that the spatial components of this equation describe acceleration under Newtonian gravity. Express the Newtonian gravitational acceleration $\mathbf{g}(x)$ in terms of the connection $\Gamma^\mu_{\nu\rho}(x)$.

Exercise 3. *Consider the geodesic deviation equation for a congruence of worldlines with unit tangents u^μ separated by dislpacement vectors s^μ:*

$$u^\rho \nabla_\rho (u^\nu \nabla_\nu s^\mu) = R^\mu{}_{\nu\rho\sigma} u^\nu u^\rho s^\sigma \tag{3}$$

In the non-relativistic limit, notice that the spatial components of this equation describe the gradient $\partial_i g_j$ of the Newtonian acceleration field $\mathbf{g}(x)$. Express this acceleration gradient in terms of the Riemann tensor $R^\mu{}_{\nu\rho\sigma}$. How is the conservative nature of Newtonian gravity encoded in the index symmetries of $R^\mu{}_{\nu\rho\sigma}$?