GENERAL RELATIVITY HOMEWORK – WEEK 7

Exercise 1. *Derive the expression for the Riemann tensor in terms of Christoffel symbols, by directly evaluating the commutator* $(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) v^\rho = R^\rho{}_{\sigma\mu\nu} v^\sigma$.

Exercise 2. *Find the components of the metric, Christoffel symbols and Riemann tensor for these two cases:*

1. *Flat 3d space in spherical coordinates* (r, θ, ϕ).
2. *A 2d sphere of radius* r*, in coordinates* (θ, ϕ).
3. *2d spacetime with the metric* $ds^2 = d\rho^2 - \rho^2 dt^2$. *What is this spacetime? What are these coordinates?*

Exercise 3. *Express the Riemann tensor* $R_{\mu\nu\rho\sigma}$ *in terms of the metric and its derivatives, working in inertial coordinates, i.e. coordinates in which* $g_{\mu\nu} = \eta_{\mu\nu}$ *and* $\partial_\mu g_{\nu\rho} = 0$ *at the point under consideration.*

Exercise 4. *Working in inertial coordinates, consider a bunch of free-falling particles moving at non-relativistic velocities. From the Newtonian point of view, each particle experiences some acceleration* a^i. *Identify the acceleration gradient* $\partial_i a^j$ *as a component of the Riemann tensor. Using the symmetries of the Riemann tensor, verify that the gravitational force is conservative.*