GENERAL RELATIVITY HOMEWORK – WEEK 4

Exercise 1. *Show that the Maxwell equations admit the following solution for the electromagnetic potential A_μ induced by charges and currents j^μ:*

$$A^\mu(x) = \int d^4y\, G(x-y)\, j^\mu(y) \ , \tag{1}$$

where $G(x)$ is the so-called retarded propagator, with support on the future lightcone:

$$G(x) = \frac{1}{2\pi}\, \delta(x_\mu x^\mu)\, \theta(x^0) \ . \tag{2}$$

The following are useful intermediate steps:

1. *Show that the potential (1) satisfies the so-called Lorenz gauge condition $\partial_\mu A^\mu = 0$.*
2. *Show that the propagator (2) satisfies $\partial_\mu \partial^\mu G(x) = 0$ for all $x^\mu \neq 0$.*
3. *Use the 4-dimensional Gauss theorem to show that precise behavior of $\partial_\mu \partial^\mu G(x)$ around $x^\mu = 0$ is $\partial_\mu \partial^\mu G(x) = \delta^4(x)$.*

Assume that everything falls off sufficiently quickly at infinity.

Exercise 2. *Consider the Euclidean plane, in Cartesian coordinates (x, y) and polar coordinates (r, θ). Write the Cartesian coordinates as functions of the polar ones, and vice versa. Apply the tensor transformation law to the Cartesian metric $g_{ij} = \delta_{ij}$, and find its components in polar coordinates. Make sure that the answer coincides with your geometric understanding. Do the same in 3d, with Cartesian coordinates (x, y, z) and spherical coordinates (r, θ, ϕ).*

Exercise 3. *Consider flat but non-orthonormal coordinates for spacetime, such that $g_{\mu\nu}$ is some constant symmetric matrix which isn't necessarily* $\mathrm{diag}(-1, 1, 1, 1)$. *In such coordinates, the surface of constant x^0 (or x^1, or x^2, or x^3) is some flat 3d hyperplane. Similarly, the "x^0 axis", i.e. the line of constant (x^1, x^2, x^3), is some straight line, and likewise for the 3 other "axes". In terms of these axes and hyperplanes, what is the geometric meaning of the metric element g_{11} being positive, negative, or zero? What is the geometric meaning of the off-diagonal element g_{12} being zero or nonzero? How about the inverse metric elements g^{11} and g^{12}?*

Exercise 4. *Consider a covector field* v_μ *and its partial derivatives* $\partial_\mu v_\nu$. *Write down the transformation of* $\partial_\mu v_\nu$ *under a general change of coordinates* $x^\mu \to x'^\mu(x^\mu)$. *Show that the antisymmetrized derivative* $\partial_\mu v_\nu - \partial_\nu v_\mu$ *transforms as a tensor.*