SPECIAL RELATIVITY HOMEWORK – WEEK 3

Consider tensors in $\mathbb{R}^3$. How many different kinds are there? Not as many as you might think. First, notice that the Hodge duality $A_{[ij]} \to \frac{1}{2}\epsilon_{ijk}A_{jk}$ can replace any antisymmetric index pair by a single index. Thus, all tensors can be reduced to totally-symmetric ones. Second, any trace part of a tensor is equivalent to a tensor with 2 fewer indices (recall the 1st week's homework). Thus, all tensors can be reduced to totally-symmetric, totally-traceless ones. That's just one independent kind of tensor for every rank n.

Exercise 1. *Let us count the independent components of a totally-symmetric, totally-traceless tensor $T_{i_1 \dots i_n}$ in $\mathbb{R}^3$.*

1. *How many independent components are in a totally-symmetric $T_{i_1 \dots i_n}$ (with no assumptions on the trace)?*
2. *How many independent components are in its trace?*
3. *How many in its totally-traceless part? Combine your answer with the 1st week's last exercise, and congratulate yourself on understanding spherical harmonics!*

Similar considerations hold for $\mathbb{R}^2$. Here, any antisymmetric index pair can be removed altogether, via the Hodge duality $A_{[ij]} \to \frac{1}{2}\epsilon_{ij}A_{ij}$. In the end, all tensors can again be reduced to the totally symmetric and traceless.

Exercise 2. *Let's discuss the independent components of a totally-symmetric, totally-traceless tensor $T_{i_1 \dots i_n}$ in $\mathbb{R}^2$.*

1. *How many independent components are in a totally-symmetric $T_{i_1 \dots i_n}$ (with no assumptions on the trace)?*
2. *How many independent components are in its trace?*
3. *How many in its totally-traceless part? Compare with the 2d analogue of spherical harmonics. (What is it, by the way?)*
4. *What is the form of these independent components (of a totally-symmetric, totally-traceless $T_{i_1 \dots i_n}$) in the null $(z, \bar{z})$ basis? How do they transform under rotation by an angle θ?*