GENERAL RELATIVITY HOMEWORK – WEEK 3

Exercise 1. *Consider a stick of length L in its rest frame, such that its endpoints stay at $x_1 = 0$ and $x_2 = L$ at all times. Now pass into a reference frame moving at velocity $-v$, so that the stick now moves at velocity v. What's the length of the stick in the new frame, defined as the distance $x'_2 - x'_1$ between its endpoints at a fixed time t'?*

Exercise 2. *Consider a gas of n particles per unit volume. Each particle has the same mass m and velocity of magnitude v. The directions of the particles' velocities are uniformly distributed. Show that the stress-energy tensor has the form:*

$$T^{\mu\nu} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \,, \tag{1}$$

and find the values of ε and P. What is the physical meaning of these components?

Exercise 3. *Now, take the limits $m \to 0$ and $v \to 1$, such that the energy E and momentum $|\mathbf{p}|$ of each particle remain finite. What happens to the length of the 4-momentum $p^\mu = (E, \mathbf{p})$? What happens to the trace T^μ_μ of the gas' stress-energy tensor? Do you understand how the two facts are related?*

Exercise 4. *In electrodynamics, the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ are just the (ti) and (ij) components of the field-strength tensor $F_{\mu\nu}$:*

$$E^i = F^{ti} \; ; \quad B^i = \frac{1}{2}\epsilon_{ijk}F^{jk} \; . \tag{2}$$

Decompose the relation $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and express $\mathbf{E}$ and $\mathbf{B}$ in terms of the potentials ϕ and $\mathbf{A}$.

Exercise 5. *The Maxwell equations in spacetime notation (in units with $\epsilon_0 = 1$) read:*

$$\partial_{[\mu}F_{\nu\rho]} = 0 \; ; \quad \partial_\nu F^{\mu\nu} = j^\mu \; . \tag{3}$$

Decompose these to find the more widely-known equations in terms of $\mathbf{E}$ and $\mathbf{B}$.

Exercise 6. *Show that the electromagnetic force formula* $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ *can be expressed in spacetime notation as:*

$$\frac{dp_\mu}{d\tau} = qF_{\mu\nu}u^\nu \ . \tag{4}$$

Note that force $\mathbf{F}$ *is always defined as momentum change per unit time* $d\mathbf{p}/dt$*.*