GENERAL RELATIVITY HOMEWORK – WEEK 2

Exercise 1. *Consider the following formulas from electromagnetism:*

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \ ; \quad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \ ; \quad \boldsymbol{\nabla} \times \mathbf{E} = -\dot{\mathbf{B}} \ ; \quad \boldsymbol{\nabla} \times \mathbf{B} = \mathbf{j} + \dot{\mathbf{E}} \ . \tag{1}$$

Rewrite these using tensor-index notation. Show that, if we describe the magnetic field as an antisymmetric matrix $B_{ij} = \epsilon_{ijk} B_k$ instead of a vector B_i, the equations can all be written without using ϵ_{ijk}. This means that, despite the high-school insistence on the right-hand rule, electromagnetism is actually reflection-invariant.

Exercise 2. *The spherical harmonics $Y_{lm}(\theta, \phi)$ can be thought of as functions $c_{i_1 \dots i_l} r_{i_1} \dots r_{i_l}$ of the unit radius-vector $r_i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, where the coefficients $c_{i_1 \dots i_l}$ are totally symmetric, totally traceless rank-l tensors. To get a flavor of how this works, consider the spherical harmonics for $l = 1, 2$. Ignoring normalization, they read:*

$$Y_{1,\pm 1} = \sin\theta \, e^{\pm i\phi} \ ; \quad Y_{10} = \cos\theta \ ; \tag{2}$$

$$Y_{2,\pm 2} = \sin^2\theta \, e^{\pm 2i\phi} \ ; \quad Y_{2,\pm 1} = \sin\theta\cos\theta \, e^{\pm i\phi} \ ; \quad Y_{20} = 3\cos^2\theta - 1 \ . \tag{3}$$

For each of the $l = 1$ harmonics, find the corresponding vector c_i. For each of the $l = 2$ harmonics, find the corresponding traceless symmetric matrix c_{ij}.