GENERAL RELATIVITY HOMEWORK – WEEK 1

Exercise 1. *Consider the motion* $\mathbf{r}(t) = (x(t), y(t), z(t))$ *of a non-relativistic object of mass* m *in the Earth's gravitational field, governed by the action:*

$$S = m \int dt \left(\frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - gz \right) \ , \tag{1}$$

assuming a constant free-fall acceleration g*.*

Find the object's trajectory, as a function of its initial position $\mathbf{r_1}$ *at time* t_1 *and its final position* $\mathbf{r_2}$ *at time* t_2*. Find the value of the action* $S(t_2, \mathbf{r_2}; t_1, \mathbf{r_1})$ *for this trajectory. Verify that the derivatives* $\partial S/\partial \mathbf{r_2}$ *coincide with the momentum* $\mathbf{p} = m\dot{\mathbf{r}}$ *at time* t_2*. Show also that* $-\partial S/\partial t_2$ *is the energy* $E = m\left(\frac{1}{2}\dot{\mathbf{r}}^2 + gz\right)$ *at* t_2*.*

Exercise 2. *In full electromagnetism, the scalar potential* $\varphi(t, \mathbf{x})$ *is joined by a vector potential* $\mathbf{A}(t, \mathbf{x})$*, and the action reads:*

$$S = \sum_n \int dt \left(\frac{m_n \dot{\mathbf{x}}_n^2}{2} + q_n(\dot{\mathbf{x}}_n \cdot \mathbf{A} - \varphi) \right) + \frac{\varepsilon_0}{2} \int dt\, d^3\mathbf{x}\, (\mathbf{E}^2 - \mathbf{B}^2) \ , \\ \text{where} \quad \mathbf{E} = -\dot{\mathbf{A}} - \boldsymbol{\nabla}\varphi \quad \text{and} \quad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \ . \tag{2}$$

This is the theory as it was known before Einstein: the EM field is secretly relativistic, but the particles still aren't.

1. *What's the canonical momentum* $\mathbf{p}_n$ *conjugate to* $\mathbf{x}_n$*? What are the Euler-Lagrange equations of motion for* $\mathbf{x}_n$*?*
2. *What are the canonical momenta* $\pi_\phi, \boldsymbol{\pi}_\mathbf{A}$ *conjugate to* φ *and* $\mathbf{A}$*? What are the Euler-Lagrange field equations for* φ *and* $\mathbf{A}$*? Express your answers in terms of* $\mathbf{E}$ *and* $\mathbf{B}$*. Hint: if you're not using index notation, remember the cyclic property of* $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$*.*
3. *What is the energy of the particles+field system? Using the field equations and integration by parts, write it as* $\frac{1}{2}\sum m_n \dot{\mathbf{x}}_n^2$ *plus a spatial integral over some energy density constructed from* $\mathbf{E}$ *and* $\mathbf{B}$*.*
4. *Now, to really test your understanding of analytical mechanics magic, derive the system's linear momentum. Again, write it as* $\sum m_n \dot{\mathbf{x}}_n$ *plus a spatial integral over a density constructed from* $\mathbf{E}$ *and* $\mathbf{B}$*. If not using index notation, remember* $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$*.*