SPECIAL RELATIVITY HOMEWORK – WEEK 1

Exercise 1. *In this exercise, we make a bit more precise the statement about commutation relations in a Lie group vs. its Lie algebra. Consider two infinitesimal symmetry transformations g_1, g_2:*

$$g_1 = 1 + \varepsilon G_1 + O(\varepsilon^2) \ ; \quad g_2 = 1 + \varepsilon G_2 + O(\varepsilon^2) \ . \tag{1}$$

Now, consider performing the transformation g_1 followed by g_2, and then trying to undo the transformations by applying their inverses in the same order. This defines the commutator $g_2^{-1} g_1^{-1} g_2 g_1$. Evaluate it to the lowest non-vanishing order in ε.

Exercise 2. *In this exercise, we further explore Galilean boosts in Hamiltonian mechanics. Consider again the Hamiltonian system of N non-relativistic particles interacting via a potential that only depends on their distances from each other:*

$$\left\{ x^i_{(n)}, p^{i'}_{(n')} \right\} = \delta^{ii'} \delta_{nn'} \ ; \tag{2}$$

$$H = \frac{1}{2} \sum_{n=1}^{N} \frac{\mathbf{p}^2_{(n)}}{m_{(n)}} + V\left(|\mathbf{x}_{(n)} - \mathbf{x}_{(n')}| \right) \ . \tag{3}$$

Here, $i = 1, 2, 3$ labels the spatial axes, and $n = 1, \dots, N$ labels the different particles. We saw that the generator of Galilean boosts is given by:

$$K^i = \sum_{n=1}^{N} \left(m_{(n)} x^i_{(n)} - p^i_{(n)} t \right) \ . \tag{4}$$

1. *Derive again the Poisson bracket $\{K^i, H\}$ of K^i with the Hamiltonian.*
2. *Verify that K^i satisfies the general time evolution formula:*

$$\frac{dK^i}{dt} = \frac{\partial K^i}{\partial t} + \{K^i, H\} \ . \tag{5}$$

3. *Now, take the opposite viewpoint on the bracket $\{K^i, H\}$, and show that it describes the effect of an infinitesimal boost on the system's energy.*
4. *Consider a moving car at velocity v, which then brakes to a stop. In this process, the car's kinetic energy transforms into heat. Now, consider the same process from a different inertial frame, which is moving with the car's initial velocity. In this frame, the car goes from rest to velocity $-v$, increasing its kinetic energy, while at the same time releasing heat! Resolve this energy-conservation paradox.*

Exercise 3. *In this exercise, we further explore the role of angular momentum as generator of rotations, and the universal form of its Poisson brackets with scalars and vectors. Consider for simplicity a single particle in 3d space. The fundamental Poisson brackets and the angular momentum read:*

$$\{x_i, p_j\} = \delta_{ij} \ ; \quad J_i = \epsilon_{ijk} x_j p_k \ . \tag{6}$$

1. *First, some degree-of-freedom counting! The phase space (x_i, p_i) is 6-dimensional. Thus, the most general observable is a function of 6 variables. Some of these observables, such as the ratio x_2/x_1 between coordinates along different axes, have no nice behavior under rotations. Some, such as $\mathbf{p}^2 = p_i p_i$, are invariant, i.e. they behave as scalars. Some, such as $\mathbf{x}^2 p_i = x_j x_j p_i$, rotate as vectors. How would you parameterize the most general observable f that behaves as a scalar? A function of how many variables is it? What about the most general vector f_i? How many functions of how many variables would you need to describe it?*

2. *Now, having parameterized most general scalar f and the most general vector f_i, verify the Poisson brackets:*

$$\{J_i, f\} = 0 \ ; \quad \{J_i, f_j\} = \epsilon_{ijk} f_k \ . \tag{7}$$