SPECIAL RELATIVITY HOMEWORK – WEEK 1

In elementary physics, there are three main "axial vectors", i.e. vectors that rely on the right-hand rule: angular velocity $\boldsymbol{\omega}$, angular momentum $\mathbf{J}$, and the magnetic field $\mathbf{B}$. As we discussed in the lecture, these quantities actually want to be bivectors, i.e. antisymmetric matrices. Let's define the relationship between the axial vector and the corresponding bivector as:

$$\omega_{ij} = \epsilon_{ijk}\omega_k \ ; \quad \omega_i = \frac{1}{2}\epsilon_{ijk}\omega_{jk} \ . \tag{1}$$

Exercise 1. *Rewrite the following equations using index notation and the bivectors ω_{ij}, J_{ij} and B_{ij}, without any ϵ_{ijk}'s in the final result.*

1. *Rotational velocity:* $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$.
2. *Angular momentum:* $\mathbf{J} = \mathbf{r} \times \mathbf{p}$.
3. *Magnetic force:* $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$.
4. *Maxwell's equations:*

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho/\epsilon_0 \ ; \quad \boldsymbol{\nabla} \times \mathbf{E} = -\dot{\mathbf{B}} \ ; \quad \boldsymbol{\nabla} \cdot \mathbf{B} = 0; \quad \boldsymbol{\nabla} \times \mathbf{B} = \mu_0(\mathbf{j} + \epsilon_0\dot{\mathbf{E}}) \ . \tag{2}$$

Exercise 2. *Do the same for the rotational inertia relation:*

$$J_i = I_{ij}\omega_j \ , \tag{3}$$

where the symmetric matrix I_{ij} is the moment of inertia:

$$I_{ij} = \sum_{particles} m(r^2\delta_{ij} - r_i r_j) \ . \tag{4}$$

Some courage may be required.