GENERAL RELATIVITY FINAL EXAM

Exercise 1. *Recall that the frame field satisfies the torsion-free condition* $De^I = 0$*. On the other hand, like any tensor-valued differential form, it satisfies* $D^2 e^I = R^{IJ} \wedge e_J$*.*

1. *Show that these facts imply a certain index symmetry of the Riemann tensor* $R_{\mu\nu\rho\sigma} = e_{\mu I} e_{\nu J} R^{IJ}_{\rho\sigma}$*.*
2. *In the non-relativistic limit of Week 7's homework, apply this symmetry to components of* $R_{\mu\nu\rho\sigma}$ *where two of the indices take the value* t*, while the two others take values among* (x, y, z)*. Which important property of the Newtonian gravitational field* $\mathbf{g}(t, \mathbf{x})$ *is encoded by the resulting identity?*

In the following two exercises, we consider a simple model of cosmology in the Cartan formalism. We denote coordinates as $\mu = t, x, y, z$, and the orthonormal basis as $I = 0, 1, 2, 3$.

Exercise 2. *A spatially-flat expanding universe is described by a frame field of the form:*

$$e^I_\mu(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a(t) & 0 & 0 \\ 0 & 0 & a(t) & 0 \\ 0 & 0 & 0 & a(t) \end{pmatrix} , \tag{1}$$

where the function $a(t)$ *is known as the scale factor.*

1. *Calculate the components of the corresponding spin-connection* ω^{IJ}_μ*.*
2. *Calculate the components of the Riemann curvature* $R^{IJ}_{\mu\nu}$*.*

Exercise 3. *Recall our discussion of 4-currents as 3-forms* $j_{\mu\nu\rho}$*. In the same way, an energy-momentum tensor can be viewed as a vector-valued 3-form* $T^I_{\mu\nu\rho}$*, which assigns 4-momentum values* p^I *to infinitesimal 3-volume elements.*

1. *Consider a spacetime of the form* (1)*, filled with massive galaxies at rest (i.e. each galaxy remains at constant spatial coordinates* x, y, z*). The galaxies' mass density is the same throughout space, and is given by;*

$$\frac{dM}{dxdydz} \equiv \rho(t) \ . \tag{2}$$

Write down the components of the corresponding energy-momentum tensor $T^I_{\mu\nu\rho}$*.*

2. *The conservation law of energy-momentum in the Cartan formalism is simply:*

$$DT^I = 0 \ . \tag{3}$$

Derive from this a condition on the function $\rho(t)$.

3. *The Einstein equations, relating curvature with energy momentum, read:*

$$\frac{1}{2}\epsilon^{IJKL} R_{JK} \wedge e_L = -8\pi G T^I \ . \tag{4}$$

Show that the ${}^{0}_{xyz}$ *component of these equations yields a differential equation for the scale factor* $a(t)$.

4. *Solve the equation for* $a(t)$*, and show that it features a Big Bang singularity.*