SPECIAL RELATIVITY – FINAL EXAM

Question 1. *Express the electromagnetic force law $m\alpha^\mu = qF^{\mu\nu}u_\nu$ in a way that makes sense also in the $m = 0$ limit.*

Question 2. *Consider the field equations of magnetostatics:*

$$\nabla \cdot \mathbf{B} = 0 \ ; \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{j} \ . \tag{1}$$

Let us now switch to spinor-index notation for 3d Euclidean space, there the vectors ∇, $\mathbf{B}$ and $\mathbf{j}$ become spinor matrices $\partial_{\alpha\beta}$, $B_{\alpha\beta}$ and $j_{\alpha\beta}$. Using this language, write a single equation that captures both equations in (1). With this as inspiration, use spinor-index notation for 4d Minkowski space to write all of Maxwell's equations as a single equation.

Question 3. *Let's play with the relativistic version of Bohr's hydrogen atom.*

a) *Consider a nucleus with Z protons, each with charge e. This generates an electrostatic field, with potential $A^t = Ze/r$. An electron with charge $-e$ and mass m is orbiting the nucleus in a classical, circular orbit. Pretend that the electron has spin 0. Find the radius r and total energy E (including potential energy), as a function of the orbit's angular momentum L. For comparison, the non-relativistic answers are:*

$$r = \frac{L^2}{mZe^2} \ ; \quad E = -\frac{mZ^2e^4}{2L^2} \tag{2}$$

b) *The ground-state orbit in Bohr's model has angular momentum $L = \hbar$. Find the largest atomic number Z for which this orbit exists.*

Question 4. *Consider now a massless spin-0 particle with charge $-q$, moving in the electrostatic potential $A^t = Q/r$ of a heavy particle with charge Q. When analyzing the motion of the massless particle, it will be useful to adopt a worldline parameter λ such that the particle's kinetic 4-momentum takes the form:*

$$p^\mu = \frac{dx^\mu}{d\lambda} \equiv \dot{x}^\mu \ . \tag{3}$$

Let's consider a general motion (not necessarily circular), and focus on the radial evolution $r(\lambda)$. It is then useful to decompose p^μ into kinetic energy $p^t = \dot{t}$, radial momentum $p^r = \dot{r}$, and tangential momentum $p^\perp = L/r$, where L is the conserved angular momentum.

a) Use energy conservation, and the fact that p^μ is lightlike, to write an equation for the radial evolution:

$$\dot{r}^2 + V(r) = E^2 \ , \tag{4}$$

where E is the total energy, and $V(r)$ is an "effective potential" that can depend on the conserved quantities E and L. Find the explicit form of $V(r)$.

b) Find the values of E and L for a circular orbit at radius r. There's no shame in extracting these from Question 3, if that's your cup of tea.

c) Now, consider a circular orbit at radius r that receives a slight, abrupt radial kick from an external force. Since the kick is radial, L remains unchanged. Since it's slight, E changes by a small amount ε. Since it's abrupt, r has no time to change during the kick, so only $\dot{r}$ changes. Express $\dot{r}$ immediately after the kick in terms of the imparted energy ε.

d) Show that after this kick, the particle will always escape to $r = \infty$. For a massless charge, circular orbits are unstable!