**GENERAL RELATIVITY FINAL EXAM: THE LIGHT WEIGHS HEAVY**

**Part I: Falling light**

**Question 1.** *In Newtonian gravity, consider a small non-relativistic particle flying at velocity $v$ past a point mass $M$, situated at $\mathbf{r} = 0$. Assume that the particle is deflected only slightly by the gravitational field of $M$. Thus, its trajectory is nearly a straight line along e.g. the $z$ axis. Let $b$ denote the "impact parameter", i.e. the trajectory's displacement from the origin along e.g. the $x$ axis. Find the small angle $\phi$ by which the gravity of $M$ deflects the particle, by integrating the $x$ component of the acceleration.*

**Question 2.** *In Special Relativity, write down the stress-energy tensor $T^{\mu\nu}$ describing a point mass $M$. Then, solve the linearized Einstein equation $\Box h_{\mu\nu} = -16\pi G(T_{\mu\nu} - \frac{1}{2}T\eta_{\mu\nu})$ in de Donder gauge for this particular $T^{\mu\nu}$, expressing the components of $h_{\mu\nu}$ in Cartesian coordinates $(t, x, y, z)$. Don't be masochistic – use 3d vector notation instead of explicit components where reasonable. Now, transform into spherical coordinates, and compare to the linearized version of the Schwarzschild solution. If different, find the linearized diffeomorphism $h_{\mu\nu} \to h_{\mu\nu} - 2\partial_{(\mu}\xi_{\nu)}$ that relates the two.*

**Question 3.** *Now, repeat Question 1, but for a massless photon moving in the linearized Schwarzschild metric of Question 2. In other words, find the photon's deflection angle by integrating the $x$ component of the geodesic equation. This was the first observational test of GR!*

**Part II: Falling into the light**

**Question 4.** *Write down the stress-energy tensor $T^{\mu\nu}$ describing a photon with energy $E$ that passes through the origin, moving in the positive $z$ direction. Express the components of $T^{\mu\nu}$ in Cartesian coordinates $(t, x, y, z)$, and also in lightcone coordinates $(u, v, x, y)$, with $t = u + v$, $z = u - v$.*

**Question 5.** *Using the Einstein equation (no linear approximations), find the stress-energy tensor that is sourcing the metric:*

$$ds^2 = -4\,dv\big(du + A(v, x, y)dv\big) + dx^2 + dy^2 \ . \tag{1}$$

*Find the function $A(v, x, y)$ that corresponds to the photon from Question 4. How many symmetries does the metric (1) with general $A(v, x, y)$ have? How many symmetries does it have with the particular $A(v, x, y)$ of a single photon?*

**Question 6.** *Now, consider a probe photon passing by, flying initially in the negative $z$ direction. As in Question 1, assume an impact parameter $b$ along the $x$ axis, and find the deflection angle due to the gravitational field of the photon from Question 5. What will happen to a probe photon that moves in the positive $z$ direction?*